1831.

EUCLID

BOOK I.

WITH NOTES AND EXERCISES

FOR THE USE OF

PREPARATORY SCHOOLS AND CANDIDATES PREPARING
FOR NAVAL CADETSHIP AND SANDHURST
PRELIMINARY EXAMINATIONS

BY

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PREFACE.

EUCLID'S proofs are adhered to in this Edition. Each proposition is set out in a form which beginners may take as a model for their written work in Euclid. A fresh line is commenced with each new step, and the usual abbreviations for words are employed, but not symbols of operation. The diagrams are large and accurately drawn, and the letters to them are varied. On the page facing each proposition the different definitions, postulates, axioms, and propositions, which are used in the construction or demonstration, are stated in full. These references are followed by Exercises which are numerous and simple, most of them being obvious deductions from the proposition to which they are attached, and similar to those set in the University Local and Previous, the Army Preliminary, the Naval Cadetship, and other Public Examinations.

Thus a boy preparing a proposition has no need to turn over leaves, but has everything clearly before him to enable him to comprehend and master his work, at the same time that his eye is being accustomed to the method in which Examiners expect propositions to be written out.

BRAITHWAITE ARNETT.

CAMBRIDGE,
January, 1885.

ABBREVIATIONS.

$\because$, because, since; $\therefore$, therefore.

=, is equal to, are equal to, be equal to.

st., straight; rt., right.

$\angle$, angle; $\angle$ s, angles.

intr., interior; extr., exterior.

$\perp$r, perpendicular; ||, parallel.

||gram, parallelogram.

$\triangle$, triangle; $\triangle$s, triangles.

sq., square; sqq., squares.

Post., Postulate; Ax., Axiom; Def., Definition.

Constr., Construction; Hyp., Hypothesis.

EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

1. A *point* is that which has no parts, or which has no magnitude.

A point is indicated by a dot, and named by a letter, as, the point *A*.

2. A *line* is length without breadth.

3. The extremities of a line are points.

The intersections of lines are points.
A line is named by two letters marking points in it, as, the line *AB*.

4. A *straight line* is that which lies evenly between its extreme points.

Straight lines are sometimes called *right* lines.
Other lines are said to be *curved*.
The *bisection* of a line is the point that divides it into two equal lines.

5. A *superficies* is that which has only length and breadth.

A superficies is also called a *surface*.
The length, breadth, and thickness of a body are called its *dimensions*.
A magnitude which has three dimensions is a *solid*.
A surface has two dimensions; a line, one dimension; a point, no dimensions, but only position.

6. The extremities of a superficies are lines.

The intersections of superficies are lines.

7. A *plane superficies* is that in which any two points being taken, the straight line between them lies wholly in that superficies.

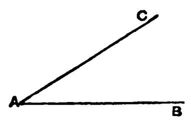
A plane superficies is generally called a *plane*.

8. A *plane angle* is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

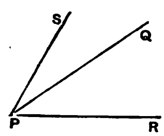
9. A *plane rectilineal angle* is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

The two lines are called the *arms* of the angle, and the point at which they meet the *angular point*, or *vertex* of the angle.

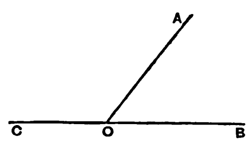
An angle is denoted (i) by a single letter at the vertex, as, the angle A, the angle at A; or (ii) by three letters, as, the angle BAC, the middle letter, A, being at the vertex, and the other two anywhere along the arms, one on each.



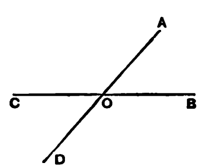
When two or more angles are at one point, each of them must be named by three letters, as, the angle QPR, the angle QPS, the angle SPR.



When a straight line AO meets another line BOC at the point O, AO is said to *make with* BOC the angles AOB, AOC; and these angles, which have a common vertex and a common arm and are on opposite sides of the common arm, are called *adjacent* angles.



When two straight lines AD, BC intersect at O, the lines are said to make with each other the angles BOA, AOC, COD, DOB; and the angles AOB, COD are called *vertical, opposite angles*, as are also the angles AOC, BOD.



The size of an angle does not depend on the length of its arms, which may be shortened or lengthened without making any change in the angle. If a straight line have one of its ends on the vertex of an angle and be turned about the vertex in the plane of the angle from the position of coinciding with one arm to that of coinciding with the other arm, the line is said to turn *through the angle*, and the angle is greater or less as the *amount of turning* is greater or less.

Angles are said to be *equal* when they could be placed on one another so that their vertices would coincide in position and their arms have the same directions.

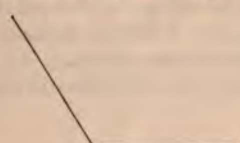
10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a *right angle;* and the straight line which stands on the other is called a *perpendicular* to it.

When one line is perpendicular, or at right angles, to another, the latter is also perpendicular, or at right angles, to the former.

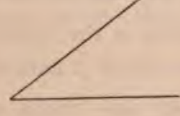
The right angle is of constant magnitude and is therefore made the standard with which all other angles are compared.

A *reflex* or *re-entrant* angle is one that is greater than two right angles.

11. An *obtuse angle* is that which is greater than a right angle.



12. An *acute angle* is that which is less than a right angle.



When two angles together equal a right angle, each is called the *complement* of the other.

When two angles together equal two right angles, each is called the *supplement* of the other.

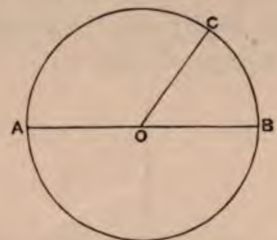
The *bisector* of an angle is the line that divides it into two equal angles.

13. A *term* or *boundary* is the extremity of any thing.

14. A *figure* is that which is enclosed by one or more boundaries.

The *area* of a plane figure is the amount of space enclosed by its boundary or boundaries.

15. A *circle* is a plane figure contained by one line, which is called the *circumference*, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.



16. And this point is called the *centre* of the circle.

A circle is named by three letters marking points on its circumference, as the circle ABC.

17. A *diameter* of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

A *radius* (plural, *radii*) of a circle is a straight line drawn from the centre to the circumference. By the definition of a circle all radii of a circle are equal to one another. The radius of a circle is half the diameter.

Any part of a circumference is called an *arc*.

The straight line which joins the extremities of an arc is called the *chord* of that arc.

18. A *semicircle* is the figure contained by a diameter and the part of the circumference cut off by the diameter.

The centre of a semicircle is the same with that of the circle.

19. A *segment of a circle* is the figure contained by a straight line and the circumference which it cuts off.

20. *Rectilineal figures* are those which are contained by straight lines.

These straight lines are called *sides;* and the sum of the sides of any rectilineal figure is called its *perimeter*.

A rectilineal figure is said to be *applied to a line* when it is constructed with this line as one of its sides.

A rectilineal figure is named by the letters which mark the angular points in order.

21. A *triangle* is a plane figure contained by three straight lines.

The side which is opposite to any angle of a triangle is said to *subtend* that angle, and the other two sides are said to *contain* or *include* the angle.

22. A *quadrilateral* is a plane figure contained by four straight lines.

A *diagonal* of a quadrilateral is a line joining two opposite angles.

A quadrilateral is sometimes named by two letters marking opposite angles.

23. A *polygon,* or *multilateral figure,* is a plane figure contained by more than four straight lines.

A polygon	with	five	sides	is called	*a pentagon.*
,,	,,	six	,,	,,	*a hexagon.*
,,	,,	seven	,,	,,	*a heptagon.*
,,	,,	eight	,,	,,	*an octagon.*
,,	,,	nine	,,	,,	*a nonagon.*
,,	,,	ten	,,	,,	*a decagon.*
,,	,,	eleven	,,	,,	*an undecagon.*
,,	,,	twelve	,,	,,	*a dodecagon.*

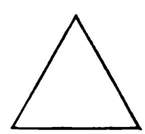
A *regular* polygon is one that has all its sides and angles equal.

A *diagonal of a polygon* is a line joining any two angles which have not a common arm.

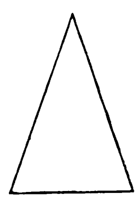
A polygon is *convex* when none of its angles are reflex or re-entrant.

Classification of triangles.

24. An *equilateral* triangle is one that has three equal sides.

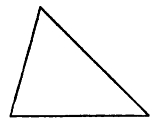


25. An *isosceles* triangle is one that has two sides equal.

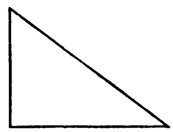


The third side is generally called the *base* of the triangle.

26. A *scalene* triangle is one that has three unequal sides.

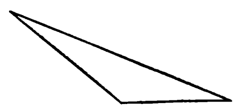


27. A *right-angled* triangle is one that has a right angle.

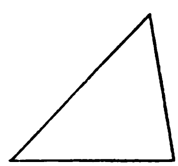


The side subtending the right angle is called the *hypotenuse*.

28. An *obtuse-angled* triangle is one that has an obtuse angle.



29. An *acute-angled* triangle is one that has three acute angles.



When two sides of a triangle have been mentioned, the third side is called the *base*.

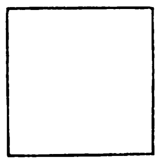
When a triangle is regarded as standing on one of its sides, this side is called its *base*, and the opposite angular point its *vertex*.

Two triangles are *equal in all respects* or *identically equal* when the three sides and the three angles of the one are respectively equal to the three sides and the three angles of the other, and the area of the one is equal to the area of the other.

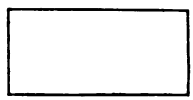
A straight line drawn from any angle of a triangle to the bisection of the opposite side is called a *median* of the triangle.

Classification of Quadrilaterals.

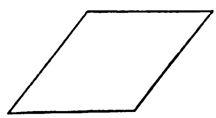
30. A *square* is a four-sided figure which has all its sides equal, and all its angles right angles.



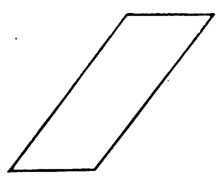
31. An *oblong* is a four-sided figure which has all its angles right angles, but not all its sides equal.



32. A *rhombus* is a four-sided figure which has all its sides equal, but its angles are not right angles.



33. A *rhomboid* is a four-sided figure which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.



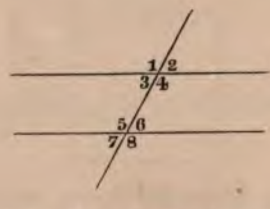
34. All other four-sided figures besides these are called *trapeziums.*

35. *Parallel* straight lines are such as are in the same plane and which being produced ever so far both ways do not meet.

A *parallelogram* is a four-sided figure which has its opposite sides parallel.

A *rectangle* is a right-angled parallelogram, and is the same figure as the oblong.

When a straight line intersects two other straight lines it makes with them eight angles, which are named as follows:



The angles marked 1, 2, 7, 8 are called *exterior* angles.
" " 3, 4, 5, 6 " *interior* "
" " 3, 6 are *alternate* angles.
" " 4, 5 " "
" " 1, 2, 3, 4 are *opposite* or *corresponding* respectively
to the " " 5, 6, 7, 8; and vice versâ.

POSTULATES.

A *postulate* is a request made by Euclid to be allowed to make an elementary construction, when needful, without any statement of the method employed.

The following are the three postulates:

1. Let it be granted that a straight line may be drawn from any one point to any other point.

2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

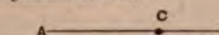
3. Let it be granted that a circle may be described from any centre, at any distance from that centre.

In making these postulates Euclid restricts himself for the purpose of constructing his figures to the use of *the straight edge of a ruler*, which must not be graduated to enable distances to be measured, and of *compasses*, which must not be employed otherwise than to describe a circle whose centre is one end of a line and whose circumference passes through the other end of the line.

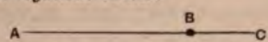
When a straight line is drawn between two fixed points which are its ends, it is called a *finite straight line.*

When a straight line may extend to any length in either direction, or in both, it is called an *indefinite straight line.*

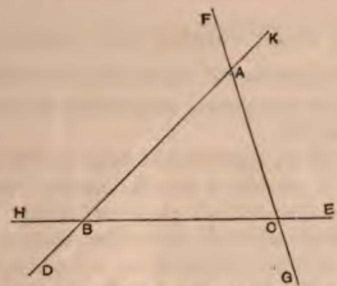
If a point C be taken in a line AB between A and B, the line AB is said to be divided *internally* in C; and AC, BC, taken from each end of the line to the *point of section* C, are called the *segments* of AB.



If the point C be taken in AB produced, the line AB is said to be *externally* divided in C; and AC, BC, taken from each end of the line AB to the *point of section* C, are still called the *segments* of AB.



When the sides of a triangle are produced both ways, there are formed at the vertices of the triangle nine additional angles, three of which are vertically opposite to the *interior* angles of the triangle. The other six are called *exterior* angles of the triangle.



Thus in the diagram, FAB, ABH, DBC, BCG, ECA, CAK are exterior angles. When one of these, as the angle ACE, is taken, the adjacent angle ACB is called the *interior adjacent* angle, and the other two interior angles CAB, ABC are called the *interior and opposite* angles.

AXIOMS.

An axiom is a statement the truth of which Euclid assumes without any demonstration of the same.

The first seven and the ninth axioms are applicable to other than geometrical magnitudes.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

The three axioms above are frequently used in Book I.

4. If equals be added to unequals the wholes are unequal.

This axiom may be divided into the two following cases, the first of which is used in Props. XVII., XXI. and XXIX.:

(i) If A and B be equal, and C be greater than D, then the sum of A and C is greater than the sum of B and D.

(ii) If A and B be equal, and C be less than D, then the sum of A and C is less than the sum of B and D.

5. If equals be taken from unequals the remainders are unequal.

This axiom is not used in Book I.

6. Things which are double of the same thing are equal to one another.

This axiom is used in Props. XXXV., XLII., XLVII.

7. Things which are halves of the same thing are equal to one another.

This axiom is used in Props. XXXVII., XXXVIII.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

This axiom as to equal magnitudes, sometimes called Euclid's *test of equality*, is used in Props. IV. and VIII.

In the application of the eighth axiom to prove two magnitudes equal, it is assumed that a magnitude, whether it be a line, an angle, or a figure, can be moved from one position to another without changing its size or form. The mental placing of one geometrical magnitude on another, such as a line on a line, or a triangle on a triangle, is called *superposition*. By this axiom if two geometrical magnitudes can be conceived to be placed, one on the other, so that their several parts and boundaries exactly coincide, the magnitudes are equal.

Figures which can be made to coincide by superposition are sometimes called *congruent* figures.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

This axiom concerning straight lines is used in Prop. IV.

11. All right angles are equal to one another.

This axiom concerning the right angle is used in Props. XIV., XV., XXVIII. and XLVII.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

This axiom, on which the propositions regarding parallel straight lines are founded, is used in Props. XXIX. and XLIV.

PROPOSITIONS.

A *proposition* is a proposal to prove or to do something.

A *corollary* to a proposition is an inference which may be deduced immediately from that proposition.

A *lemma* is a proposition which is demonstrated for the purpose of giving assistance in the demonstration of another proposition.

There are two kinds of propositions:—Theorems and Problems.

A *theorem* is a proposition in which is demonstrated the truth of certain statements of the properties of specified lines, angles, or figures.

A theorem consists of two parts:—the *hypothesis* (supposition), that which is assumed, or granted: and the *conclusion*, that which is stated to follow from the hypothesis.

Q. E. D. at the end of a theorem is an abbreviation of *Quod erat demonstrandum*, which was to be proved.

A theorem is said to be the *converse* of another theorem when the hypothesis of each is the conclusion of the other. The converse of a true proposition is not necessarily true.

The *obverse* of a theorem is formed by writing negatively the hypothesis and conclusion of the original theorem.

Thus, if the original theorem be:

If A is B, then C is D; ....(i)

its converse is:

If C is D, then A is B; ....(ii)

its obverse is:

If A is not B, then C is not D; ....(iii)

and the obverse of its converse
or the converse of its obverse is:

If C is not D, then A is not B. ....(iv)

Theorem (iv) is called the ***contrapositive*** of (i).

Thus also (iii) is the contrapositive of (ii).

If of these theorems any two, which are not contrapositive of one another, are true, it follows that the rest are true.

For example, take the theorem:

If two sides of a triangle be equal, the opposite angles are equal. I. v.

Its obverse is:

The greater side of every triangle has the greater angle opposite to it. I. XVIII.

Hence the converse theorem must follow:

If two angles of a triangle be equal, the subtending sides are equal. I. VI.

And the converse of the obverse:

The greater angle of every triangle is subtended by the greater side. I. XIX.

Propositions IV. and XXIX. may be taken as other examples in Book I.

The propositions in the First Book of Euclid's Elements may be divided into three sections.

Section I., embracing Propositions I.—XXVI., is devoted chiefly to the Properties of Triangles.

Section II. contains Propositions XXVII.—XXXIV. and treats of Parallel Straight Lines.

Section III., which includes Propositions XXXV.—XLVIII., relates to Equality of Area in Figures, which do not necessarily coincide after superposition.

THEOREMS IN SECTION I.

ORIGINAL THEOREM.		CONVERSE THEOREM.	
HYPOTHESIS.	CONCLUSION.	HYPOTHESIS.	CONCLUSION.
STRAIGHT LINES AND ANGLES.			
COROLLARY TO PROP. XI.			
Suppose it possible that two straight lines can have a common segment;	it shall be impossible.		
PROPOSITION XIII.		PROPOSITION XIV.	
If one straight line makes with another straight line on one side two angles;	these angles either shall be two right angles or together equal to two right angles.	If, at a point in a straight line, two other straight lines on the opposite sides of it make the adjacent angles together equal to two right angles;	these two straight lines shall be in one and the same straight line.
PROPOSITION XV.			
If two straight lines cut one another;	the vertical, opposite angles shall be equal.		
COROLLARY 1.			
If two straight lines cut one another;	the angles, which they make at the point where they cut, are together equal to four right angles.		
COROLLARY 2.			
If any number of straight lines meet at a point;	all the angles made by them are together equal to four right angles.		

PROPERTIES OF A SINGLE TRIANGLE.

PROPOSITION V.

If a triangle have two sides equal, and if the equal sides be produced; the angles at the base shall be equal, and the angles on the other side of the base shall be equal to one another.

COROLLARY.

If a triangle be equilateral; it is equiangular.

PROPOSITION XVI.

If there be a triangle and one side be produced; the exterior angle shall be greater than either of the interior opposite angles.

PROPOSITION XVIII.

If a triangle have one side greater than another; the greater side shall have the greater angle opposite to it.

PROPOSITION XX.

If there be any triangle; any two of its sides shall be together greater than the third side.

PROPOSITION XXI.

If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle; these shall be less than the other two sides of the triangle, but shall contain a greater angle.

PROPOSITION VI.

If two angles of a triangle be equal to one another; the sides which subtend the equal angles shall be equal to one another.

COROLLARY.

If a triangle be equiangular; it is equilateral.

PROPOSITION XVII.

(Converse of Axiom 12.)

If there be any triangle; any two of its angles shall be together less than two right angles.

PROPOSITION XIX.

If a triangle have one angle greater than another; the greater angle shall have the greater side opposite to it.

THEOREMS IN SECTION I. (*continued*).

EQUALITY OF TWO TRIANGLES.

ORIGINAL THEOREM. Hypothesis.	ORIGINAL THEOREM. Conclusion.	CONVERSE THEOREM. Hypothesis.	CONVERSE THEOREM. Conclusion.
		PROPOSITION VII. (Lemma to Prop. VIII.)	
		Suppose it possible that on the same base and on the same side of it, there can be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another;	it shall be impossible.
PROPOSITION IV.		**PROPOSITION VIII.**	
If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal;	they shall have their bases equal, and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.	If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal;	the angle contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other.
PROPOSITION XXVI.			
If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each;	then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.		

INEQUALITY OF TWO TRIANGLES.

PROPOSITION XXIV.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle shall be greater than the base of the other.

PROPOSITION XXV.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; the angle contained by the two sides of that which has the greater base shall be greater than the angle contained by the sides equal to them, of the other.

THEOREMS IN SECTION II.

PARALLELS.

PROPOSITION XXVII.

If a straight line falling on two other straight lines, make the alternate angles equal to one another; the two straight lines shall be parallel to one another.

PROPOSITION XXVIII.

If a straight line falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side together equal to two right angles; the two straight lines shall be parallel to one another.

PROPOSITION XXIX.

If a straight line fall on two parallel straight lines; it shall make the alternate angles equal to one another; and shall make the exterior angle equal to the interior and opposite angle on the same side, and also the two interior angles on the same side together equal to two right angles.

PROPOSITION XXX.

If straight lines be parallel to the same straight line; they shall be parallel to each other.

THEOREMS IN SECTION II. (*continued*).

ORIGINAL THEOREM.		CONVERSE THEOREM.	
HYPOTHESIS.	CONCLUSION.	HYPOTHESIS.	CONCLUSION.
PROPOSITION XXXII.			
If there be a triangle, and one side be produced;	the exterior angle shall be equal to the two interior and opposite angles, and the three interior angles shall be together equal to two right angles.		
COROLLARY 1.			
If there be any rectilineal figure;	all its interior angles, together with four right angles, shall be equal to twice as many right angles as the figure has sides.		
COROLLARY 2.			
If there be any rectilineal figure;	the exterior angles shall be together equal to four right angles.		
PROPOSITION XXXIII.			
If straight lines join the extremities of two equal and parallel straight lines towards the same parts;	they shall also themselves be equal and parallel.		
PROPOSITION XXXIV.			
If there be a parallelogram;	its opposite sides and angles shall be equal to one another, and the diameter shall bisect it.		

THEOREMS IN SECTION III.

EQUALITY OF FIGURES IN RESPECT OF AREA.

PROPOSITION XXXV.

If parallelograms be on the same base and between the same parallels;	they shall be equal to one another.

PROPOSITION XXXVI.

If parallelograms be on equal bases and between the same parallels;	they shall be equal to one another.

PROPOSITION XXXVII.

If triangles be on the same base and between the same parallels;	they shall be equal to one another.

PROPOSITION XXXVIII.

If triangles be on equal bases, and between the same parallels;	they shall be equal to one another.

PROPOSITION XLI.

If a parallelogram and a triangle be on the same base and between the same parallels;	the parallelogram shall be double of the triangle.

PROPOSITION XXXIX.

If triangles be equal, and be on the same base and on the same side of it;	they shall be between the same parallels.

PROPOSITION XL.

If triangles be equal, and be on equal bases in the same straight line, and on the same side of it;	they shall be between the same parallels.

THEOREMS IN SECTION III. (*continued.*)

ORIGINAL THEOREM.		CONVERSE THEOREM.	
HYPOTHESIS.	CONCLUSION.	HYPOTHESIS.	CONCLUSION.
EQUALITY OF FIGURES IN RESPECT OF AREA.			
PROPOSITION XLIII.			
If through any point in the diagonal of a parallelogram straight lines be drawn parallel to the sides, thus forming two parallelograms about the diameter and two other parallelograms called complements;	the complements shall be equal to one another.		
PROPOSITION XLVII.		PROPOSITION XLVIII.	
If a triangle be right-angled;	the square described on the side subtending the right angle shall be equal to the squares on the sides which contain the right angle.	If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it;	the angle contained by these two sides shall be a right angle.
THE RECTANGLE.			
COROLLARY TO PROP. XLVI.			
If a parallelogram has one right angle;	all its angles are right angles.		

A *problem* is a proposition in which is determined the method by which lines may be drawn, angles made, or figures described, having specified properties.

A problem consists of two parts:—the data, the things given: and the quæsita, the things required to be done.

Q. E. F. at the end of a problem is an abbreviation of *Quod erat faciendum*, which was to be done.

PROBLEMS IN SECTION I.

DATA.	QUÆSITA.
PROPOSITION I.	
Given a finite straight line,	describe an equilateral triangle on it.
PROPOSITION II.	
Given a point and a straight line,	from the point draw a straight line equal to the line.
PROPOSITION III.	
Given two straight lines, one greater than the other,	cut off from the greater a part equal to the less.
PROPOSITION IX.	
Given a rectilineal angle,	bisect it.
PROPOSITION X.	
Given a finite straight line,	bisect it.
PROPOSITION XI.	
Given a straight line and a point in it,	from the point draw a straight line at right angles to the line.
PROPOSITION XII.	
Given a straight line of unlimited length, and a point without it,	from the point draw a straight line perpendicular to the line.
PROPOSITION XXII.	
Given three straight lines, any two whatever of which are greater than the third,	describe a triangle having its sides respectively equal to the three lines.
PROPOSITION XXIII.	
Given a straight line and a point in it, and a rectilineal angle,	at the point in the line make an angle equal to the angle.

PROBLEMS IN SECTIONS II. AND III.

DATA.	QUÆSITA.
PROPOSITION XXXI.	
Given a point and a straight line,	through the point draw a straight line parallel to the line.
PROPOSITION XLII.	
Given a triangle, and a rectilineal angle,	describe a parallelogram equal to the triangle and having an angle equal to the angle.
PROPOSITION XLIV.	
Given a straight line, a triangle, and a rectilineal angle,	to the line apply a parallelogram equal to the triangle and having an angle equal to the angle.
PROPOSITION XLV.	
Given a rectilineal figure, and a rectilineal angle,	describe a parallelogram equal to the figure and having an angle equal to the angle.
COROLLARY.	
Given a straight line, a rectilineal figure, and a rectilineal angle,	to the line apply a parallelogram equal to the figure and having an angle equal to the angle.
PROPOSITION XLVI.	
Given a straight line,	describe a square on it.

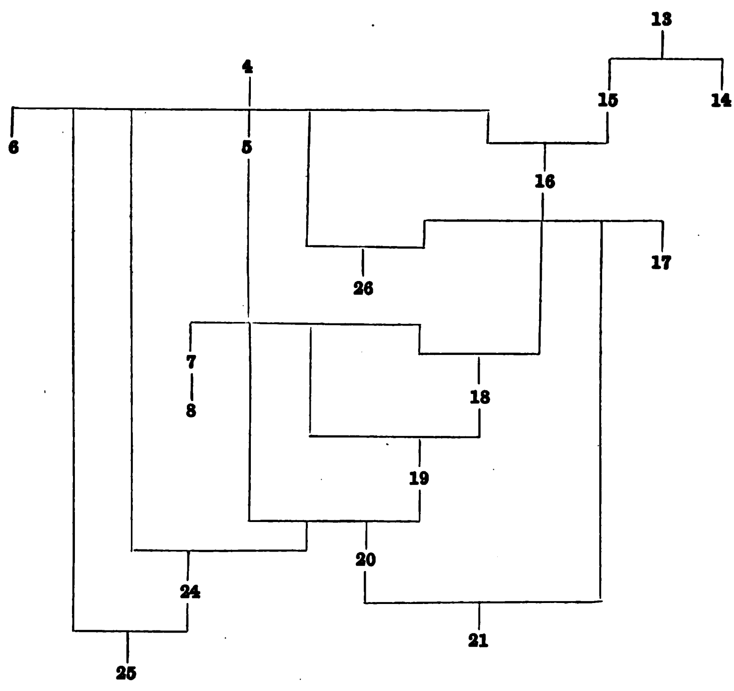
The *enunciation* of a proposition is the general statement of what is supposed or given, and of what has to be proved or done.

The *demonstration* of a proposition consists of successive steps of reasoning by which it is shewn in a theorem that the conclusion follows from the hypothesis; and, in a problem that the construction effects what was proposed to be done.

When a proposition has been proved by the *direct* method of demonstration from results previously assumed or established, its converse is usually demonstrated by Euclid *indirectly*. Proposition XLVIII, however, is an exception. The indirect method of demonstration, called the *reductio ad absurdum*, is one in which a statement is proved to be true by its being shewn that the assumption of the falsity of the statement leads to conclusions that are known to be untrue. Vide Propositions VI, VIII, XIV, XIX, XXV, XXIX, XXXIX, XL. Propositions which take the form of a denial that certain properties or constructions are possible must of necessity be proved indirectly, as Proposition VII.

CHART OF THEOREMS IN SECTION I.

SHEWING THE PROPOSITIONS REQUIRED IN THE DEMONSTRATION OF EACH.



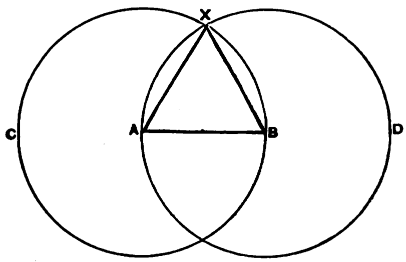
For example, to prove Proposition XXIV there are used Props. IV, V, and XIX. No previous proposition is employed in the proof of Proposition IV. Proposition VI is not required by Euclid in Section I.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given st. line.

It is required to describe an equilateral $\triangle$ on AB.



From the centre A, at the distance AB, describe the circle BXC. *Post.* 3.

From the centre B, at the distance BA, describe the circle AXD. *Post.* 3.

From the point X, in which the circles cut one another, draw the st. lines XA, XB, to the points A, B. *Post.* 1.

Then ABX shall be an equilateral $\triangle$.

$\because$ the point A is the centre of the circle BXC,

$\therefore AX = AB$. *Def.* 15.

And $\because$ the point B is the centre of the circle AXD,

$\therefore BX = AB$. *Def.* 15.

But it has been shewn that $AX = AB$;

$\therefore AX, BX$ each $= AB$.

But things which are equal to the same thing are equal to one another;

$\therefore AX = BX$; *Ax.* 1.

$\therefore AB, BX, XA$ are equal to one another.

Wherefore *the* $\triangle$ ABX *is equilateral, and it is described on the given st. line* AB. *Def.* 24.

Q. E. F.

REFERENCES.

Def. 15. A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

Def. 24. An equilateral triangle is one that has three equal sides.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Post. 3. Let it be granted that a circle may be described from any centre, at any distance from that centre.

EXERCISES.

1. What is the perimeter of the equilateral triangle ABC, when $BC = 5$ ft.?

2. If AB in the figure of Prop. I. be produced both ways to meet the circles at C and D, and an equilateral triangle CDZ be described on CD, how many times will the perimeter of the triangle CDZ contain AB?

3. With the same figure shew that if Y be the other point in which the circles BXC, AXD cut one another, the quadrilateral $AXBY$ is equilateral.

4. An equilateral triangle ABC is described on BC, one of the sides of a square $BCDE$; prove that the pentagon $ABEDC$ is equilateral.

5. Using the figure of Prop. I, describe a circle with centre X and radius XA and prove that this circle will pass through B; also if it cut the circle BXC at P and the circle AXD at Q, shew that the triangles PAX, QBX will be equilateral.

6. Find a straight line equal to the perimeter of a triangle.

7. Produce a straight line PQ to R, so that QR may be equal to PQ; also shew how to find a straight line five times the length of a given straight line.

8. From the extremity Q of a straight line PQ draw three other straight lines each equal to PQ.

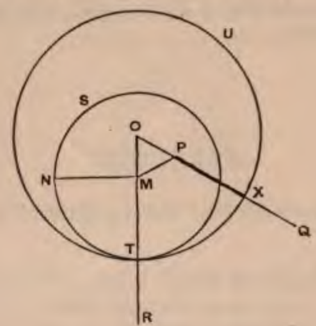
9. Upon a given base describe an isosceles triangle, having each of the equal sides double of the base.

PROPOSITION II. PROBLEM.

From a given point to draw a straight line equal to a given straight line.

Let P be the given point, and MN the given st. line.

It is required to draw from the point P a st. line equal to MN.



From the point P to M draw the st. line PM, *Post.* 1.

upon PM describe the equilateral $\triangle\ PMO$, I. I.

and produce the st. lines OP, OM to Q and R. *Post.* 2.

From the centre M, at the distance MN, describe the circle NST, cutting OR in the point T, *Post.* 3.

and, from the centre O, at the distance OT, describe the circle TUX, cutting OQ in the point X. *Post.* 3.

Then the st. line PX shall $= MN$.

$\because$ the point M is the centre of the circle NST,

$\therefore MN = MT$, *Def.* 15.

and $\because$ the point O is the centre of the circle TUX,

$\therefore OX = OT$, *Def.* 15.

and OP, OM, parts of them, are equal; *Def.* 24.

$\therefore$ the remainder $PX =$ the remainder MT. *Ax.* 3.

But it has been shewn that $MN = MT$;

$\therefore PX$, MN each $= MT$.

But things which are equal to the same thing are equal to one another;

$\therefore PX = MN$. *Ax.* 1.

Wherefore *from the given point P a st. line PX has been drawn equal to the given st. line MN.*

Q. E. F.

REFERENCES.

Def. 15. A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

Def. 24. An equilateral triangle is one that has three equal sides.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

Post. 3. Let it be granted that a circle may be described from any centre, at any distance from that centre.

Ax. 3. If equals be taken from equals, the remainders are equal.

I. I. To describe an equilateral triangle on a given finite straight line.

EXERCISES.

1. C is the centre of two circles PQR, MNO; radii CP, CQ of the larger circle cut the circumference of the smaller circle at M, N; shew that MP is equal to NQ.

2. If the given point in Prop. II. be joined with the other end of the given straight line, and the equilateral triangle be described (*a*) on one side, (*b*) on the other side of this line, if also the sides of the equilateral triangle be produced through the vertex as well as beyond the base, draw four figures shewing the constructions, and give the proof in each case.

3. In OP produced take any point Q, and from Q draw a straight line equal to OP.

4. Solve the problem, Prop. II, when the given point is the vertex of an equilateral triangle described on the given straight line.

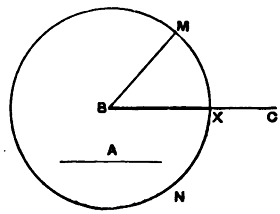
5. On the given base MN describe an isosceles triangle having each of the sides equal to a given straight line, greater than the half of MN.

PROPOSITION III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let A and BC be the 2 given st. lines, of which BC is the greater.

It is required to cut off from BC, the greater, a part equal to A, the less.



From the point B draw the st. line BM equal to A, I. II.
and from the centre B, at the distance BM, describe the circle MXN, cutting BC in X. *Post.* 3.

Then BX shall $= A$.

$\because$ B is the centre of the circle MXN,

$\therefore BX = BM$. *Def.* 15.

But $A = BM$, *Constr.*

$\therefore BX, A$ each $= BM$,

$\therefore BX = A$. *Ax.* 1.

Wherefore *from BC the greater of two given st. lines a part BX has been cut off equal to A the less.*

Q. E. F.

REFERENCES.

Def. 15. A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

Post. 3. Let it be granted that a circle may be described from any centre, at any distance from that centre.

Ax. 1. Things which are equal to the same thing are equal to one another.

I. II. From a given point to draw a straight line equal to a given straight line.

EXERCISES.

1. Draw a straight line equal to the difference of two given straight lines.

2. Produce the less of two given straight lines so that it may be equal to the greater.

3. Draw a straight line equal to the sum of two given straight lines.

4. $PQRS$ is a straight line, and PQ is equal to RS; shew that PR is equal to QS.

5. In a straight line $MNOPQ$, MO is equal to NP, NP to OQ, and OP to PQ; shew that N is the middle point of MO.

6. The sum and difference of two lines are together double of the greater of them.

7. AB is a straight line produced to C, so that BC is equal to AB; in BC any point P is taken; shew that twice BP is equal to the difference of AP and PC.

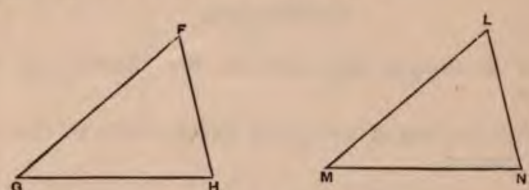
8. A straight line AB is produced to D, and from BD a part BC is cut off equal to AB; in CD any point P is taken; shew that BP is equal to half the sum of AP and PC.

PROPOSITION IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely, those to which the equal sides are opposite.

Let FGH, LMN be two $\triangle$s, which have the two sides FG, FH equal to the two sides LM, LN, each to each, namely, FG to LM, and FH to LN, and the $\angle\, GFH$ equal to the $\angle\, MLN$.

Then the base GH shall $=$ the base MN; and the $\triangle\ FGH$ shall $=$ the $\triangle\ LMN$; and the other $\angle$s shall be equal, each to each, to which the equal sides are opposite, namely, the $\angle\, FGH$ to the $\angle\, LMN$, and the $\angle\, FHG$ to the $\angle\, LNM$.



For, if the $\triangle\ FGH$ be applied to the $\triangle\ LMN$,
so that the point F may be on L, and the st. line FG on LM;
then the point G will coincide with the point M,
$\because FG = LM$; *Hyp.*
And, FG coinciding with LM,
FH will fall on LN,
$\because$ the $\angle\, GFH =$ the $\angle\, MLN$; *Hyp.*
$\therefore$ also the point H will coincide with the point N,
$\because FH = LN$. *Hyp.*
But the point G was shewn to coincide with the point M;
$\therefore$ the base GH will coincide with the base MN.
$\because$ G coinciding with M, and H with N,
if the base GH does not coincide with the base MN,
two straight lines will enclose a space, which is impossible. *Ax.* 10.
$\therefore$ the base GH coincides with the base MN, and $=$ it. *Ax.* 8.
$\therefore$ the whole $\triangle FGH$ coincides with the $\triangle LMN$, and $=$ it, *Ax.* 8.
and the other $\angle$s of the one coincide with the other $\angle$s of the other, and are equal to them,
namely, the $\angle\, FGH$ to the $\angle\, LMN$,
and the $\angle\, FHG$ to the $\angle\, LNM$.

Wherefore *if two triangles &c.* Q. E. D.

REFERENCES.

Ax. 8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

Ax. 10. Two straight lines cannot enclose a space.

EXERCISES.

1. The sides AB, AD of the quadrilateral $ABCD$ are equal, and the diagonal AC bisects the angle BAD. Shew that BC is equal to CD, and that AC bisects the angle BCD.

2. The diagonals of a square are equal.

3. $ABCD$ is a square and E, F, G, H the middle points of the sides; shew that the figure $EFGH$ is equilateral.

4. ABC, DEF, GHK are three triangles. Each of the sides AB, DF, HK is 6 inches long, and each of the sides AC, EF, GH, 4 inches long; also each of the angles BAC, DEF, GHK is a right angle. Which of the triangles are equal in all respects?

5. The angle BAC of a scalene triangle ABC, which has AB greater than AC, is bisected by AD meeting the base at D. From AB cut off AE equal to AC, join DE, and prove that DE is equal to DC, and that the angle EDC is bisected by AD.

6. The straight line which bisects the vertical angle of an isosceles triangle bisects the base at right angles.

7. Two triangles equal in all respects and having a common side lie on opposite sides of it; prove that the line joining their vertices is either perpendicular to the common side, or else divides the figure formed by the triangles into equal parts.

8. From any point E in the straight line CD two equal straight lines EA, EB, are drawn making equal angles with ED, and CA, DA, CB, DB are joined; prove that the triangles ACD, BCD are equal in all respects.

9. Prove Prop. IV. beginning the superposition at G instead of at F.

10. If two squares have a side of one equal to a side of the other, prove by superposition that the squares are equal in all respects.

[Hence, any square whose side is equal to a given straight line is called *the square on that line.*]

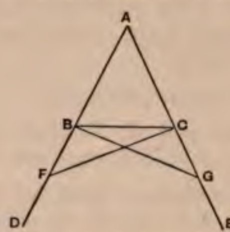
11. Two quadrilaterals have three sides of the one equal to three sides of the other, each to each, and the angles contained by equal sides equal, each to each; prove by superposition that the quadrilaterals are equal in all respects.

12. In the two pentagons $ABCDE$, $FGHKL$, AB, BC, CD, DE are respectively equal to FG, GH, HK, KL, and the angles ABC, BCD, CDE are respectively equal to the angles FGH, GHK, HKL; shew by superposition that the pentagons are equal in all respects.

PROPOSITION V. THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

Let ABC be an isosceles $\triangle$, of which the side $AB =$ the side AC, and let the equal sides AB, AC be produced to D and E.
Then shall the $\angle\, ABC =$ the $\angle\, ACB$, and the $\angle\, CBD =$ the $\angle\, BCE$.



In BD take any point F;
from AE the greater cut off AG equal to AF the less, I. III.
and join FC, GB. *Post.* 1.
$\because AF = AG$, and $AC = AB$, *Constr.*
the 2 sides FA, $AC =$ the 2 sides GA, AB, each to each, and they contain the $\angle\, FAG$ common to the 2 $\triangle$s AFC, AGB;
$\therefore$ the base $FC =$ the base GB, I. IV.
the $\triangle\ AFC =$ the $\triangle\ AGB$, and
the remaining $\angle$ s of the one = the remaining $\angle$ s of the other, each to each, to which the equal sides are opposite,
namely, the $\angle\ ACF$ to the $\angle\ ABG$,
and the $\angle\ AFC$ to the $\angle\ AGB$.
And $\because$ the whole $AF =$ the whole AG, *Constr.*
of which the part $AB =$ the part AC; *Hyp.*
$\therefore$ the remainder $BF =$ the remainder CG, *Ax.* 3.
and FC was shewn $= GB$;
$\therefore$ the 2 sides BF, $FC =$ the 2 sides CG, GB, each to each,
and the $\angle\ BFC$ was shewn $=$ the $\angle\ CGB$;
$\therefore$ the $\triangle$s BFC, CGB are equal, I. IV.
and their other $\angle$ s are equal, each to each, to which the equal sides are opposite,
namely, the $\angle\ FBC$ to the $\angle\ GCB$,
and the $\angle\ BCF$ to the $\angle\ CBG$.
And $\because$ it has been shewn that the whole $\angle$ s ABG, ACF are equal, and that the parts of these, the $\angle$s CBG, BCF, are equal;
$\therefore$ the remaining $\angle\ ABC =$ the remaining $\angle\ ACB$, *Ax.* 3.
which are the $\angle$s at the base of the $\triangle\ ABC$.
And it has also been shewn that the $\angle\ FBC =$ the $\angle\ GCB$, which are the $\angle$s on the other side of the base.

Wherefore *the angles &c.* Q. E. D.

Corollary. Hence every equilateral triangle is also equi-*angular.*

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 3. If equals be taken from equals the remainders are equal.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

EXERCISES.

1. An equilateral triangle NMO is described on NO, a side of the square $NOPQ$; shew that the angle MNQ is equal to the angle MOP.

2. A square $ABCD$ and a rhombus $ABEF$ have a side AB common; prove that the angle BCE is equal to the angle BEC.

3. The opposite angles of a rhombus are equal.

4. ACB, ADB are two isosceles triangles on the same base and on the same side of it; prove that the angle between CA and DA, both produced, is equal to the angle between CB and DB, both produced.

5. A quadrilateral $ABCD$ has the sides AB, AD equal to one another, and also the sides BC, CD equal; prove that the angles ABC, ADC are equal to one another.

6. ABC is a scalene triangle. On AB, produced if necessary, take AD equal to AC, and similarly on AC take AE equal to AB. Join ED, DC, and prove that the triangles CED, CEB are equal in all respects.

7. The base MN of an isosceles triangle OMN is produced both ways to P and Q, so that MP is equal to NQ. Prove that POQ is an isosceles triangle.

8. The lines joining the middle points of the equal sides of an isosceles triangle to the opposite angles will be equal to one another.

9. PQR is an equilateral triangle; in PQ take PM equal to one-fourth PQ and in QR take QN equal to one-fourth QR; shew that PN is equal to MR.

10. If in the three sides PQ, QR, RP of an equilateral triangle, distances PM, QN, RO be taken, each equal to one-third of a side, shew that MNO is equilateral.

11. XYZ is an isosceles triangle having XY equal to XZ; in YZ take any point P, and from ZY cut off ZQ equal to YP; shew that XP is equal to XQ.

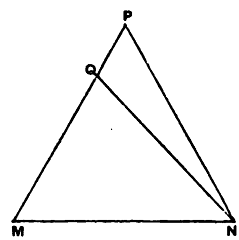
12. BAC is an equilateral triangle on the same base as the triangle BDC, which is isosceles, the point A being within the triangle BDC. Shew that if AD be joined, the triangles ABD, ACD are identically equal.

PROPOSITION VI. THEOREM.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

Let PMN be a $\triangle$ having the $\angle PMN$ equal to the $\angle PNM$.

Then the side PM shall $=$ the side PN.



For if PM be not $= PN$,

one of them must be greater than the other.

Let PM be the greater, and from it cut off QM equal to PN, the less, I. III.

and join QN. *Post.* 1.

Then in the $\triangle$s QMN, PNM,

$\because QM = PN$, *Constr.*

and MN is common to both;

the 2 sides QM, $MN =$ the 2 sides PN, NM, each to each,

and the $\angle QMN =$ the $\angle PNM$; *Hyp.*

$\therefore$ the base $QN =$ the base PM,

and the $\triangle QMN =$ the $\triangle PNM$, I. IV.

the less to the greater, which is absurd. *Ax.* 9.

$\therefore$ PM is not unequal to PN,

that is, $PM = PN$.

Wherefore *if two angles &c.* Q. E. D.

Corollary. Hence every equiangular triangle is also equilateral.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 9. The whole is greater than its part.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

EXERCISES.

1. $OATB$ is a four-sided figure of which the side OA is equal to OB, and the angles OAT, OBT are right angles; prove that TA is equal to TB.

2. $ABCD$ is a quadrilateral having the side AB equal to AD, and the angle ABC to the angle ADC; shew that CB is equal to CD.

3. In PQ, the base of an isosceles triangle OPQ, there is taken a point M, and the line MN, which meets OQ in N, is such that the angle NMQ is equal to the angle OPQ. Shew that NM is equal to NQ.

4. In the figure of Prop. v, if FC and BG meet at H, shew that FH and GH are equal.

5. In the same figure, if FK be drawn at right angles to FC and equal to it, and GL be drawn at right angles to BG and equal to it, and if BK and CL, or these lines produced, meet at O; shew that OB is equal to OC, and that OKL is also an isosceles triangle.

6. $BCGHF$ is a pentagon having the angle FBC equal to the angle BCG, and the sides BF, FH equal to the sides CG, GH, each to each; shew that, if BG and CF intersect in O, the triangles OBC, HBC are isosceles; also prove that if FB, GC be produced to meet at A, then the triangle ABC is isosceles.

7. PQR is a triangle having the angle PQR equal to the angle PRQ; in QR take any point M and from RQ cut off RN equal to QM. Prove that the angle PMQ is equal to the angle PNR.

8. Prove by "*reductio ad absurdum*" that, if two sides of a triangle be unequal, the angles which they subtend shall also be unequal; and, if two angles of a triangle be unequal, the sides which subtend the unequal angles shall also be unequal.

PROPOSITION VII. THEOREM.

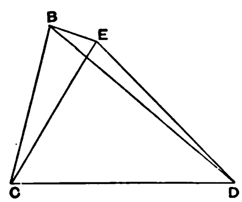
(LEMMA TO PROP. VIII.)

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.

If it be possible, on the same base CD, and on the same side of it, let there be two $\triangle$s CBD, CED, having their sides BC, EC, which are terminated at the extremity C of the base, equal to one another, and likewise their sides BD, ED, which are terminated at B, equal to one another.

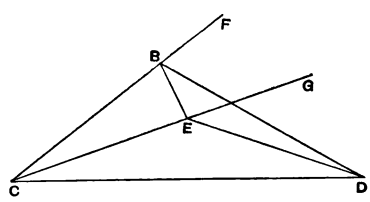
Join BE. *Post.* 1.

First, let the vertex of each $\triangle$ be without the other $\triangle$.



$\because CB = CE$, *Hyp.*
the $\angle CBE =$ the $\angle CEB$. I. v.
But the $\angle CBE$ is greater than the $\angle DBE$; *Ax.* 9.
$\therefore$ also the $\angle CEB$ is greater than the $\angle DBE$;
much more then is the $\angle DEB$ greater than the $\angle DBE$.
Again, $\because DB = DE$, *Hyp.*
the $\angle DEB =$ the $\angle DBE$. I. v.
But it has been shewn to be greater; which is impossible.

Next, let the vertex E of one $\triangle$ be within the other $\triangle$ CBD, and produce CB, CE to F, G. *Post.* 2.



$\because CB = CE$, in the $\triangle CBE$, *Hyp.*
the $\angle$s FBE, GEB, on the other side of the base, are equal to one another. I. v.

But the $\angle FBE$ is greater than the $\angle DBE$; *Ax.* 9.
$\therefore$ also the $\angle GEB$ is greater than the $\angle DBE$;
much more then is the $\angle DEB$ greater than the $\angle DBE$.
Again, $\because DB = DE$, *Hyp.*
the $\angle DEB =$ the $\angle DBE$. I. v.
But it has been shewn to be greater; which is impossible.

The case in which the vertex of one $\triangle$ is on a side of the other, needs no demonstration.

Wherefore *on the same base &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

Ax. 9. The whole is greater than its part.

I. v. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

EXERCISES.

1. Draw the figure for the third case and explain why no demonstration is needed.

2. From the vertex O of the angle POR, OS is drawn such that the angle POS is equal to the angle SOR, and OQ is any line drawn from O between OS and OR; prove that the angle POQ is greater than the angle QOR.

3. $ABCD$ is a quadrilateral, and the diagonals AC, BD intersect at E; if the angle CBA is equal to the angle CAB, prove that the angle DAB is greater than the angle DBA.

4. On the same base, and on the same side of it, there cannot be two isosceles triangles whose equal sides are each equal to a given straight line and whose vertices do not coincide with one another.

5. On the same straight line, and on the same side of it, there cannot be two equilateral triangles not coinciding with one another.

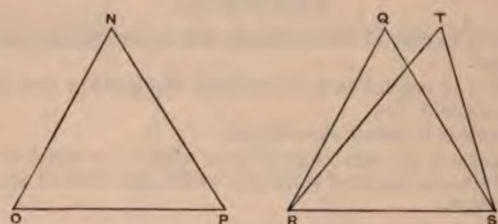
6. Two circles whose centres are A and B cannot intersect in two points, both on the same side of the straight line AB.

PROPOSITION VIII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

Let NOP, QRS be two $\triangle$s, having the two sides NO, NP equal to the two sides QR, QS, each to each, namely, NO to QR, and NP to QS, and also the base OP equal to the base RS.

Then the $\angle ONP$ shall $=$ the $\angle RQS$.



For, if the $\triangle NOP$ be applied to the $\triangle QRS$,
so that the point O may be on R, and the st. line OP on RS;
the point P will coincide with the point S,
$\because OP = RS$; *Hyp.*
$\therefore OP$ coinciding with RS,
ON and NP will coincide with RQ and QS.
For, if the base OP coincides with the base RS,
and the sides ON, NP do not coincide with the sides RQ, QS,
but have a different situation as RT, TS;
then, on the same base and on the same side of it there will be two $\triangle$s having their sides which are terminated at one extremity of the base equal to one another, and likewise their sides which are terminated at the other extremity.
But this is impossible. I. VII.
$\therefore$ the base OP coinciding with the base RS,
the sides ON, NP must coincide with the sides RQ, QS.
$\therefore$ also the $\angle ONP$ coincides with the $\angle RQS$, and $=$ it. *Ax.* 8.
Wherefore *if two triangles &c.* Q. E. D.

Since the triangles coincide, they are seen to be equal in all respects, but Euclid does not state this for, it being now proved that the triangles have two sides and the included angles equal, it follows that their areas and their remaining angles are equal by Prop. IV.

REFERENCES.

Ax. 8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

I. VII. On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.

EXERCISES.

1. The opposite angles of a rhombus are equal, and its diagonals bisect the angles through which they pass.

2. A rhombus is bisected by its diagonal.

3. In a circle equal chords subtend equal angles at the centre.

4. At $32\frac{8}{11}$ minutes after 3 o'clock the distance between the points of the hands of a watch is the same as it was at 3. Shew that the hands are at right angles.

5. *ABC* is an isosceles triangle, having *AB* equal to *AC*. In *AB* take any point *D*, and from *AC* cut off *AE* equal to *AD*, and join *BE*, *CD*, cutting each other at *F*. Shew that *DEF* is an isosceles triangle, and that *AF* bisects the angle *BAC*.

6. *ABCD* is a quadrilateral, having *AB* equal to *CD*, and *AD* to *BC*; shew that the opposite angles are equal.

7. Two equal straight lines *AB* and *CD* are joined towards opposite parts by the equal straight lines *AD*, *CB*, intersecting in *O*; prove that the triangles *OAC*, *OBD* are isosceles.

8. If two isosceles triangles be on the same base, the straight line joining their vertices, or that straight line produced, will bisect the base at right angles, and also bisect the vertical angles.

9. If two circumferences of circles cut each other, the line joining the points of intersection will be bisected at right angles by the line joining the centres.

10. From every point of a given straight line, the straight lines drawn to each of two given points on opposite sides of the line are equal; prove that the line joining the given points will be bisected at right angles by the given line.

11. The diagonals of a square bisect each other at right angles, and divide the square into four equal triangles.

12. The diagonals of a rhombus bisect each other at right angles, and divide the rhombus into four equal triangles.

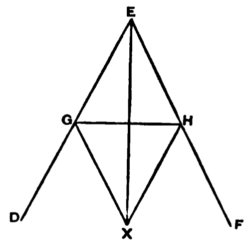
13. If *AB*, *AC* be the equal sides of an isosceles triangle, and if their middle points be the centres of two equal circles intersecting in *D*, prove that *AD* bisects the angle *BAC*.

14. The triangles *ABC*, *DEF* have *AB* equal to *DE*, the angle *ABC* equal to the angle *DEF*, and also their areas equal; prove by superposition that they are equal in all respects.

PROPOSITION IX. PROBLEM.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let DEF be the given rectilineal $\angle$.
It is required to bisect it.



Take any point G in ED,
and from EF cut off EH equal to EG; I. III.
join GH; *Post.* 1.
on GH, on the side remote from E, describe the equilateral $\triangle\, GHX$, I. I.
and join EX. *Post.* 1.
The st. line EX shall bisect the $\angle\, DEF$.
$\because EG = EH$, *Constr.*
and EX is common to the two $\triangle$s GEX, HEX,
the 2 sides GE, EX = the 2 sides HE, EX, each to each,
and the base GX = the base HX; *Def.* 24.
$\therefore$ the $\angle\, GEX$ = the $\angle\, HEX$. I. VIII.
Wherefore *the given rectilineal $\angle\, DEF$ is bisected by the st. line EX.* Q. E. F.

If the triangle GHX were described on the *same* side of GH as E is, and not on the side *remote* from it, as in the above construction, the point X might coincide with E, and then the bisector EX could not be drawn.

REFERENCES.

Def. 24. An equilateral triangle is one that has three equal sides.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

I. I. To describe an equilateral triangle on a given finite straight line.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. VIII. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

EXERCISES.

1. In the figure, shew that the angle GXH and the straight line GH are also bisected by EX.

2. Divide an angle of an equilateral triangle into four equal parts.

3. Divide a given angle into two angles such that one of them may be seven times the other.

4. Divide a given obtuse angle into two parts such that one part may be fifteen times the other.

5. Divide a given angle into two angles such that a third of one of them may be equal to a fifth of the other.

6. Shew that the angle between the bisectors of two adjacent complementary angles is half a right angle.

7. The angle BAC is bisected by AE, and AD is a straight line drawn outside the angle BAC; shew that the angle EAD is equal to half the sum of the angles BAD, CAD; and that the angle EAC is equal to half the difference of the same angles.

8. Find two angles whose sum is a right angle, and whose difference is half the angle of an equilateral triangle.

9. Let AB, AC be equal sides of the triangle ABC; bisect the angles ABC, BCA by the straight lines BD, CD, meeting at D, and prove that BDC is an isosceles triangle.

10. BAC is a triangle having the angle B double of the angle A. Bisect the angle B by BD meeting AC at D, and shew that BD is equal to AD.

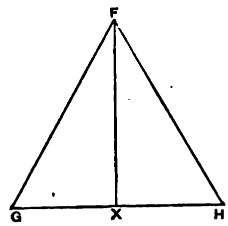
11. Prove by bisecting an angle of an equilateral triangle that if one of the acute angles of a right-angled triangle be double of the other, the hypotenuse is double of the side subtending the least angle.

12. Construct an isosceles triangle whose vertical angle is half the angle of an equilateral triangle, and each of whose equal sides is equal to a given straight line.

PROPOSITION X. PROBLEM.

To bisect a given finite straight line, that is, to divide it into equal parts.

Let GH be the given finite st. line.
It is required to bisect GH.



On GH describe the equilateral $\triangle$ GHF, I. I.
and bisect the $\angle$ GFH by the st. line FX, meeting GH at X. I. IX.

Then GH shall be bisected in the point X.

$\because$ $GF = HF$, *Def.* 24.
and FX is common to the two $\triangle$s GFX, HFX,
the 2 sides GF, FX = the 2 sides HF, FX, each to each,
and the $\angle$ GFX = the $\angle$ HFX; *Constr.*
$\therefore$ the base GX = the base HX. I. IV.

Wherefore *the given st. line GH is bisected in the point X.*

Q. E. F.

The figures of Propositions IX., X., XI., XII. are examples of what are called *symmetrical* constructions.

A figure is said to be *symmetrical with regard to a line*, called the *axis of symmetry*, when any straight line drawn at right angles to the line and terminated by the boundary of the figure is bisected by the line. The axes of symmetry in the figures mentioned are the lines EX, FX, GX, and PX.

Two points are symmetrical with regard to a straight line, when this line bisects at right angles the line joining the points.

REFERENCES.

Def. 24. An equilateral triangle is one that has three equal sides.

I. I. To describe an equilateral triangle on a given finite straight line.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. IX. To bisect a given rectilineal angle.

EXERCISES.

1. ABC is a given straight line, AB being greater than BC; divide AB in D, so that the difference between AD and DB may be equal to BC.

2. Divide a given straight line into eight equal parts.

3. Divide a given straight line into two parts such that one part may be three times the other.

4. Divide a given straight line into two parts, such that three times one part may be equal to five times the other.

5. Find two straight lines equal respectively to half the sum and half the difference of two given straight lines.

6. Find two lines whose sum and difference are given.

7. Produce a given straight line to such a point that twice the whole line thus produced may be equal to five times the given line.

8. If, with the extremities of a given straight line AB as centres and any line greater than the half of AB as radius, two circles be described intersecting in X and Y, shew that XY will bisect AB at right angles.

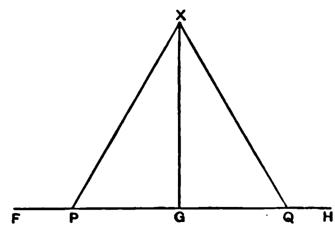
9. Construct a right-angled triangle having the hypotenuse double one of the sides.

10. Describe an isosceles triangle, having given the perimeter and the base.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

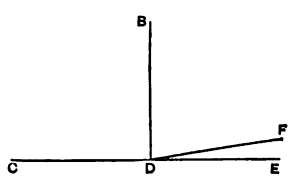
Let FH be the given st. line, and G the given point in it.
It is required to draw from the point G a st. line at rt. $\angle$s to FH.



Take any point P in GF, and make GQ equal to GP. I. III.
On PQ describe the equilateral $\triangle PQX$, I. I.
and join GX. *Post.* 1.
Then GX drawn from the point G shall be at rt. $\angle$s to FH.
$\because PG = QG$, *Constr.*
and GX is common to the two $\triangle$s PGX, QGX,
the 2 sides PG, GX = the 2 sides QG, GX, each to each,
and the base PX = the base QX; *Def.* 24.
$\therefore$ the $\angle PGX$ = the $\angle QGX$; I. VIII.
and they are adjacent $\angle$s.
But when a st. line, standing on another st. line, makes the adjacent $\angle$s equal to one another, each of the $\angle$s is called a rt. $\angle$. *Def.* 10.
$\therefore$ each of the $\angle$s PGX, QGX is a rt. $\angle$.
Wherefore *from the given point G in the given st. line FH, GX has been drawn at rt. $\angle$s to FH.* Q. E. F.

Corollary. By the help of this problem it may be shewn that two st. lines cannot have a common segment.

If it be possible, let the two st. lines CDE, CDF have the segment CD common to both of them.



From the point D draw DB at rt. $\angle$s to CD. I. XI.

Then, $\because$ CDE is a st. line, *Hyp.*
$\therefore$ the $\angle BDE =$ the $\angle CDB$. *Def.* 10.
Again, $\because$ CDF is a st. line, *Hyp.*
$\therefore$ the $\angle BDF$ also $=$ the $\angle CDB$. *Def.* 10.
$\therefore$ the $\angle BDF =$ the $\angle BDE$, *Ax.* 1.
the less to the greater; which is impossible. *Ax.* 9.

Wherefore *two straight lines cannot have a common segment.*

REFERENCES.

Def. 10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle.

Def. 24. An equilateral triangle is one that has three equal sides.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 9. The whole is greater than its part.

I. I. To describe an equilateral triangle on a given finite straight line.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. VIII. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other.

EXERCISES.

1. Shew that this proposition is a particular case of Prop. IX.

2. Draw a straight line at right angles to a given straight line from one extremity of the line.

3. From the same point in a given straight line there can be drawn in a plane on one side of the line, only one straight line at right angles to the given line.

4. At a given point A in a given straight line AB make an angle BAC equal to half a right angle.

5. At a given point O in the straight line OP make an angle POQ equal to three-fourths of a right angle.

6. If from the middle point D of a straight line BC, DA be drawn at right angles to BC, and P be a point such that PB is unequal to PC, prove that P cannot be in DA.

7. Find a point in a straight line such that its distances from two given points may be equal. Draw figures for all the different cases.

8. Describe a circle which shall pass through two given points, and have its centre in a given straight line.

9. Find a point equidistant from the angular points of an equilateral triangle.

10. Having given the diagonals, describe a rhombus.

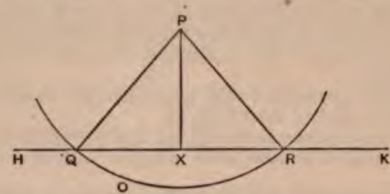
11. Describe a right-angled triangle, having given one of the sides containing the right angle, and also the perpendicular from the right angle on the hypotenuse.

PROPOSITION XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let HK be the given st. line which may be produced to any length both ways, and let P be the given point without it.

It is required to draw from the point P a st. line ⊥r to HK.



On the other side of HK take any point O, and from the centre P at the distance PO describe the circle QOR, meeting HK at Q and R. *Post.* 3.

Bisect QR at X, I. x.

and join PX. *Post.* 1.

Then PX drawn from the point P shall be ⊥r to HK.

Join PQ, PR.

$\because QX = RX$, *Constr.*

and XP is common to the two $\triangle$s QXP, RXP,

the 2 sides QX, XP = the 2 sides RX, XP, each to each,

and the base PQ = the base PR; *Def.* 15.

$\therefore$ the $\angle QXP$ = the $\angle RXP$; I. VIII.

and they are adjacent $\angle$s.

But when a st. line, standing on another st. line, makes the adjacent $\angle$s equal to one another, each of the $\angle$s is called a rt. $\angle$, and the st. line which stands on the other is called a ⊥r to it. *Def.* 10.

Wherefore *a ⊥r PX has been drawn to the given st. line HK from the given point P without it.* Q. E. F.

The straight line is described as of unlimited length to admit of its being produced long enough to meet the circle.

The point X is called the *foot* of the perpendicular PX.

REFERENCES.

Def. 15. A circle is a plane figure contained by one line, which is called the circumference, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Post. 3. Let it be granted that a circle may be described from any centre, at any distance from that centre.

I. VIII. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other.

I. X. To bisect a given finite straight line.

EXERCISES.

1. From the angular points of a triangle perpendiculars are drawn to the opposite sides, and produced so that the parts produced are equal to the perpendiculars. The extremities of these produced lines are joined to the angular points of the triangle. Prove that three triangles are thus formed equal in all respects to the original triangle.

2. Shew that a circle cannot cut a straight line in more than two points.

3. If two straight lines bisect each other at right angles, any point in either of them is equidistant from the extremities of the other.

4. A is any point without the straight line BC. Draw AB perpendicular to BC, and produce it to D, making BD equal to AB. If C be any point in BC, prove that ACD is an isosceles triangle and that CB bisects the angle ACD.

5. Through two given points on opposite sides of a given straight line draw two straight lines which shall meet in that given straight line and include an angle bisected by that given straight line.

6. The straight line drawn from the vertex of an isosceles triangle to the middle point of the base is perpendicular to the base, and bisects the vertical angle.

7. On a given straight line as base describe an isosceles triangle having the perpendicular from the vertex on the base equal to a given straight line.

8. If there be two triangles satisfying the conditions of Prop. VII except that they are on opposite sides of the base, shew that the line joining their vertices is perpendicular to the base.

PROPOSITION XIII. THEOREM.

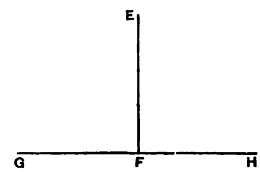
The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

Let the st. line EF make with the st. line GH, on one side of it, the $\angle$s EFG, EFH.

These are either two rt. $\angle$s, or together = two rt. $\angle$s.

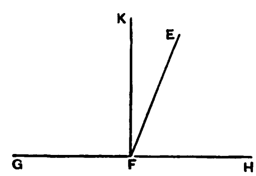
For, if the $\angle EFG$ = the $\angle EFH$,

each of them is a rt. $\angle$. *Def.* 10.



But, if the $\angle EFG$ be not = the $\angle EFH$,

from F draw FK at rt. $\angle$s to GH. I. XI.



Then the $\angle$s GFK, KFH are two rt. $\angle$s. *Def.* 10.

Now $\because$ the $\angle HFK$ = the two $\angle$s HFE, EFK,

to each of these equals add the $\angle KFG$;

$\therefore$ the $\angle$s HFK, KFG = the 3 $\angle$s HFE, EFK, KFG. *Ax.* 2.

Again, $\because$ the $\angle GFE$ = the 2 $\angle$s GFK, KFE,

to each of these equals add the $\angle EFH$;

$\therefore$ the $\angle$s GFE, EFH = the 3 $\angle$s GFK, KFE, EFH. *Ax.* 2.

But the $\angle$s HFK, KFG have been proved = the same 3 $\angle$s;

$\therefore$ the $\angle$s GFE, EFH = the $\angle$s HFK, KFG. *Ax.* 1.

But HFK, KFG are 2 rt. $\angle$s;

$\therefore$ GFE, EFH together = 2 rt. $\angle$s. *Ax.* 1.

Wherefore *the angles &c.* Q. E. D.

REFERENCES.

Def. 10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 2. If equals be added to equals, the wholes are equal.

I. XI. To draw a straight line at right angles to a given straight line, from a given point in the same.

EXERCISES.

1. Shew that the angle *EFK* in the figure is half the difference of the angles *EFG*, *EFH*.

2. *O* is a point in the straight line *POQ*. Shew that all the successive angles, *POA*, *AOB*, &c. made by any number of straight lines drawn from *O* on the same side of *PQ* are together equal to two right angles.

3. On *BC* a side of the triangle *ABC* is described a square *BCDE*, and *ABE* is a straight line. Shew that the triangle is right-angled.

4. If two exterior angles of a triangle, made by producing one side both ways, be equal to one another, the triangle is isosceles.

5. If the two sides *AB*, *AC* of the triangle *ABC* be produced to *D* and *E*, and the exterior angles *DBC*, *ECB* be equal to each other, the triangle is isosceles.

6. When two unequal angles are supplementary, the greater must be obtuse and the less acute. Hence, of the angles, which the bisector of the vertical angle of any triangle makes with the base, the acute angle is subtended by the less side.

7. The greater of two supplementary angles is five times the less; what fraction of a right angle is the less?

8. The two straight lines which bisect an exterior angle of a triangle and its adjacent interior angle are at right angles to each other.

9. Let the side *BC* of the triangle *ABC* be produced both ways to *D* and *E*; shew that the difference of the angles *ABC*, *BCA* is equal to the difference of the angles *ACE*, *ABD*.

10. From the point *O* in the straight line *MON*, the two straight lines *OP*, *OQ* are drawn on the same side of *MN*; if the angle *PON* be equal to the angle *QOM*, and the angle *POQ* be bisected by *OR*, shew that *OR* will be at right angles to *MN*.

11. Two squares *ABCD*, *EFGH* are placed with the points *B* and *E* coincident, and the sides *AB*, *EF* in the same straight line: also the squares are on the same side of *AF*; shew that *BC* must fall on *EH*.

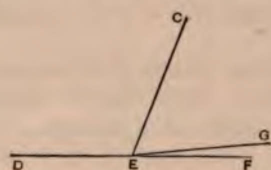
12. From *O* the middle point of the straight line *GOE* a straight line *OC* is drawn equal to *OG* or *OE*; the straight lines *CG*, *CE* are bisected at *F* and *D*, and *OF*, *OD* are joined. Prove that *FOD* is a right angle.

PROPOSITION XIV. THEOREM.

If, at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point E in the st. line CE, let the two st. lines ED, EF, on the opposite sides of CE, make the adjacent $\angle$s CED, CEF together $=$ 2 rt. $\angle$s.

EF shall be in the same st. line with DE.



For, if EF be not in the same st. line with DE,
if possible, let EG be in the same st. line with it.

Then, $\because$ CE makes with the st. line DEG, on one side of it, the $\angle$s CED, CEG,

these $\angle$s together $=$ two rt. $\angle$s. I. XIII.

But the $\angle$s CED, CEF together $=$ two rt. $\angle$s; *Hyp.*

$\therefore$ the $\angle$s CED, $CEG =$ the $\angle$s CED, CEF. {*Ax.* 11. *Ax.* 1.

From each of these equals take away the common $\angle$ CED;

$\therefore$ the remaining $\angle$ $CEG =$ the remaining $\angle$ CEF, *Ax.* 3.

the less to the greater; which is impossible. *Ax.* 9.

$\therefore$ EG is not in the same st. line with DE.

And in the same manner it may be shewn that no other can be in the same st. line with it but EF;

$\therefore$ EF is in the same st. line with DE.

Wherefore *if, at a point &c.* Q. E. D.

To prove that a straight line PQ passes through a point O.

Join OP, OQ; and shew that OP, OQ are in the same straight line.

REFERENCES.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 3. If equals be taken from equals the remainders are equal.

Ax. 9. The whole is greater than its part.

Ax. 11. All right angles are equal to one another.

I. XIII. The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

EXERCISES.

1. Three angles POQ, QOR, ROS are each equal to two-thirds of a right angle. Shew that POS is a straight line.

2. PRT is a triangle in which PRT is a right angle, and on RT is described a square $RTSQ$; prove that PR, RQ are in the same straight line.

3. Two triangles have two sides of the one equal to two sides of the other, each to each; and the angles contained by the equal sides are together equal to two right angles. The triangles are so placed that a side of one coincides with the side equal to it of the other. Prove that the remaining equal sides are in the same straight line.

4. The right angle POQ is divided into two parts by the straight line ON. The angle POR is equal to the angle PON, and the angle QOS to the angle QON. Shew that ROS is a straight line.

5. AOB is a right angle, and C any point; draw CD, CF perpendiculars to OA, OB, and produce them to E and G, making DE equal to CD, and FG equal to FC; shew that the straight line GE passes through O.

6. ABC is a triangle having the angle ABC greater than the angle CAB, and AB is produced to D. The angle CAE, on the side of CA opposite to AB, exceeds the angle CBD by as much as the angle ABC exceeds the angle CAB; shew that AE is in the same straight line with AB.

7. PQR is a straight line, and from the point Q on opposite sides of PR are drawn two straight lines QS, QT such that the angles SQR, PQT are equal to each other. Prove that QT is in the same straight line with QS.

8. On AC, CB, the sides containing the right angle C in the triangle ABC, are described two isosceles triangles ACD, CBE, having the angles at the bases each half a right angle; DCE is a straight line.

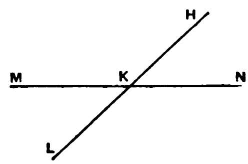
9. The angle BAC of the triangle ABC is a right angle. On AB, AC are described externally the squares $ABDE$, $ACFG$. Prove that the straight line DF passes through A.

10. If two squares $KLMN$, $OPQR$ be placed so that L coincides with O, and LM falls on OR, then KL must either fall on OP or be in one and the same straight line with it.

PROPOSITION XV. THEOREM.

If two straight lines cut one another, the vertical, opposite angles shall be equal.

Let the two st. lines HL, MN cut one another at the point K. Then shall the $\angle HKM =$ the $\angle LKN$, and the $\angle MKL =$ the $\angle HKN$.



∵ the st. line MK makes with the st. line HKL the $\angle$s HKM, MKL,

these $\angle$s together = two rt. $\angle$s. I. XIII.

Again, ∵ the st. line LK makes with the st. line MKN the $\angle$s MKL, LKN,

these $\angle$s together also = two rt. $\angle$s. I. XIII.

But it has been shewn that the $\angle$s HKM, MKL together = two rt. $\angle$s;

∴ the $\angle$s HKM, $MKL =$ the $\angle$s MKL, LKN. $\left\{\begin{array}{r} Ax.\ 11. \\ Ax.\ 1. \end{array}\right.$

From each of these equals take away the common $\angle MKL$.

∴ the remaining $\angle HKM =$ the remaining $\angle LKN$. *Ax.* 3.

In the same manner it may be shewn that

the $\angle MKL =$ the $\angle HKN$.

Wherefore *if two straight lines, &c.* Q. E. D.

Corollary 1. From this it is manifest that, if two st. lines cut one another, the $\angle$s which they make at the point where they cut, together = four rt. $\angle$s.

Corollary 2. And consequently, that all the $\angle$s made by any number of st. lines meeting at one point, together = four rt. $\angle$s.

REFERENCES.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 3. If equals be taken from equals the remainders are equal.

Ax. 11. All right angles are equal to one another.

I. XIII. The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

EXERCISES.

1. Prove the converse of this proposition:

If four straight lines meet at a point so that the opposite angles are equal, these straight lines are two and two in the same straight line.

2. Let two straight lines AB, CD cut one another at E. Bisect the angle AEC by EF, and the angle BED by EG. Shew that EF, EG are in the same straight line.

3. The four bisectors of the angles which one straight line makes with another straight line intersecting it form two straight lines at right angles to one another.

4. The opposite sides of the quadrilateral formed by joining the extremities of two diameters of a circle are equal.

5. If the diagonals AC, DB of a quadrilateral $ABCD$ bisect one another, the opposite sides of the quadrilateral are equal.

6. Prove that the ends of the beam of a balance will move to positions equidistant from their original positions, when unequal weights are placed in the scales.

7. Two trains passed a viaduct, one over it and the other under it, at 9 a.m. Supposing that the lines are straight and that uniform speeds were maintained, shew that the trains were at the same distance from each other at 9. 10 as they were at 8. 50.

8. If two straight lines intersect at a point, and one of the vertical angles is a right angle, the other three are right angles.

9. O is the middle point of NP, a side of the triangle MNP, and MO is produced to Q so that OQ is equal to MO; prove that NQ is equal to MP.

N.B. *This construction is often useful in riders, especially when the middle points of the sides of a triangle are introduced.*

10. From two given points on the same side of a given line draw two lines which shall meet in that line and make equal angles with it.

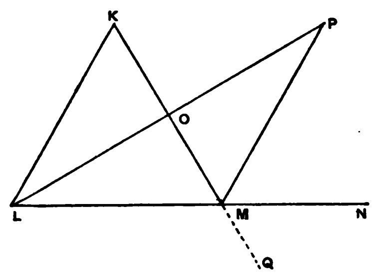
11. If a straight line which bisects the vertical angle of a triangle also bisects the base, the triangle is isosceles.

PROPOSITION XVI. THEOREM.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let KLM be a $\triangle$, and let one side LM be produced to N.

The extr. $\angle\ KMN$ shall be greater than either of the intr. opposite $\angle$ s MKL, KLM.



Bisect KM at O; I. x.

join LO, and produce it to P, making OP equal to LO, { *Post.* 1. *Post.* 2. I. III.

and join PM. *Post.* 1.

$\because KO = OM$, and $OL = OP$, *Constr.*

the 2 sides KO, $OL =$ the 2 sides MO, OP, each to each,

and the $\angle\ KOL =$ the $\angle\ MOP$,

$\because$ they are opposite vertical $\angle$ s; I. xv.

$\therefore$ the $\triangle\ KOL =$ the $\triangle\ MOP$, I. iv.

and the remaining $\angle$ s $=$ the remaining $\angle$ s, each to each, to which the equal sides are opposite;

$\therefore$ the $\angle\ OKL =$ the $\angle\ OMP$.

But the $\angle\ OMN$ is greater than the $\angle\ OMP$, *Ax.* 9.

$\therefore$ KMN is greater than the $\angle\ MKL$.

In the same manner, if LM be bisected, and the side KM be produced to Q, it may be shewn that the $\angle\ LMQ$, that is the $\angle\ KMN$, is greater than the $\angle\ KLM$. I. xv.

Wherefore *if one side &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

Ax. 9. The whole is greater than its part.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. X. To bisect a given finite straight line.

I. XV. If two straight lines cut one another, the vertical, opposite angles shall be equal.

EXERCISES.

1. If the straight lines which bisect the angles of a triangle also bisect the opposite sides, the triangle is equilateral.

2. Prove fully the second part of Prop. XVI.

3. Is the exterior angle of a triangle necessarily greater than its interior adjacent angle? Illustrate your answer by figures.

4. The base BC of an isosceles triangle is produced to D, and AD is joined; prove that the angle ABD is greater than the angle ADB.

5. ABC is a triangle having the angle BAC bisected by the straight line AD; if CD be equal to CA, and any point P be taken in AD, prove that the angle CPA is greater than the angle BAD.

6. Take any point D in BC, a side of the triangle ABC; join AD, and shew that the angles ABC, ACB are together less than two right angles.

7. The angle CBD of the triangle BCD is bisected by BE, which meets CD at E; prove that the angle CEB is greater than the angle CBE, and the angle BED greater than the angle DBE.

8. The two exterior angles of a triangle, made by producing a side both ways, are together greater than two right angles.

9. The two sides AB, AC of the triangle ABC are produced to D and E; shew that the exterior angles DBC, BCE are together greater than two right angles.

10. In the figure to Prop. XVI., prove the following properties, in each case indicating the triangle and the side which is produced:

(a) The angle KLO is less than each of the angles KOP, LOM;

(b) The angle LKM is less than each of the angles KOP, LOM, KMN, LMQ;

(c) The angle OLM is less than each of the angles KMN, LMQ, KOL, MOP.

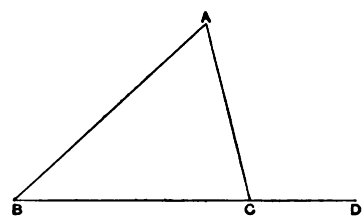
Also prove that the area and the sum of the interior angles of the triangle PLM are respectively equal to the area and the sum of the interior angles of the triangle KLM.

PROPOSITION XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be a $\triangle$.

Any two of its $\angle$ s are together less than two rt. $\angle$ s.



Produce BC to D. *Post.* 2.

Then $\because$ ACD is the extr. $\angle$ of the $\triangle ABC$,

it is greater than the intr. opposite $\angle ABC$. I. XVI.

To each of these add the $\angle BCA$.

$\therefore$ the $\angle$ s ACD, ACB are greater than the $\angle$ s ABC, BCA.

But the $\angle$ s ACD, ACB together = two rt. $\angle$ s; I. XIII.

$\therefore$ the $\angle$ s ABC, BCA are together less than two rt. $\angle$ s.

In the same manner it may be shewn that

the $\angle$ s BCA, CAB are together less than two rt. $\angle$ s,

as also the $\angle$ s CAB, ABC.

Wherefore *any two angles &c.* Q. E. D.

This proposition is afterwards included in the more general theorem, Prop. XXXII.

REFERENCES.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

I. XIII. The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

I. XVI. If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

EXERCISES.

1. No triangle can have both a right angle and an obtuse angle.

2. No triangle can have more than one right angle or one obtuse angle.

3. Of the three angles of a triangle two must be acute.

4. The three angles of a triangle are together less than three right angles.

5. The three exterior angles of a triangle made by producing the sides successively in the same direction are together greater than three right angles.

6. If two angles of a triangle be unequal, the less is acute.

7. In a right-angled triangle, the right angle is the greatest angle.

8. In an obtuse-angled triangle, the obtuse angle is the greatest angle.

9. From the same point there can be drawn to a given straight line only one perpendicular.

10. The angles at the base of an isosceles triangle are acute.

11. An equilateral triangle is an acute-angled triangle.

12. The bisectors of any two angles of a triangle will meet.

13. In the figure, Prop. XVI., shew that LK, MP, if produced ever so far, cannot meet to form a triangle with KM.

14. From the point O in the straight line MON, the line OP is drawn making the angle MOP obtuse, and consequently NOP acute. From any point P in OP a perpendicular is drawn to MN. Prove that the foot of the perpendicular must be in ON.

15. The perpendicular from the vertex of a triangle on the base falls within the triangle, if the vertical angle be obtuse or be a right angle; or, if the triangle be an acute-angled triangle.

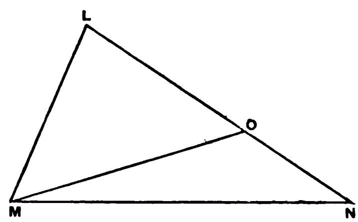
16. The perpendicular from either of the acute angles of an obtuse-angled triangle on the opposite side falls without the triangle.

PROPOSITION XVIII. THEOREM.

The greater side of every triangle has the greater angle opposite to it.

Let LMN be a $\triangle$, of which the side LN is greater than the side LM.

The $\angle LMN$ is also greater than the $\angle LNM$.



$\because$ LN is greater than LM,	*Hyp.*
make LO equal to LM, and join MO.	I. III. *Post.* 1.
Then $\because$ LOM is the extr. $\angle$ of the $\triangle MON$,	
it is greater than the intr. opposite $\angle ONM$.	I. XVI.
But the $\angle LOM =$ the $\angle LMO$,	I. V.
$\because$ the side $LO =$ the side LM.	*Constr.*
$\therefore$ the $\angle LMO$ is also greater than the $\angle LNM$.	
Much more then is the $\angle LMN$ greater than the $\angle LNM$.	*Ax.* 9.

Wherefore *the greater side &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 9. The whole is greater than its part.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. V. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

I. XVI. If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

EXERCISES.

1. All the angles of a scalene triangle are unequal.

2. The angle opposite to the less of two unequal sides of a triangle must be acute.

3. The sides AB, BC, CA of a triangle ABC are respectively 5, 10, 7; arrange the angles in order of magnitude.

4. The base BC of an isosceles triangle is 10, and the perimeter is 26; which is the greatest angle?

5. A friend's house is 4 miles due E. from a small town, which lies to the N.W. from my house. The distance between our two places is 3 miles, and my friend says his house is due N. from mine. Shew that his judgment is incorrect.

[The angle between N. and N.W., as also the angle between E. and S.E., is half a right angle.]

6. If the bisector of an angle of a triangle divide the opposite side into unequal segments, the sides containing the angle shall be unequal.

7. $OPQR$ is a quadrilateral figure of which OR is the longest side, and PQ the shortest side; shew that the angle OPQ is greater than the angle ORQ, and the angle PQR greater than the angle POR.

8. In any triangle the perpendicular from the opposite angle on a side which is not less than either of the remaining sides falls within the triangle.

9. $ABCD$ is a quadrilateral of which the diagonal BD bisects the angle ABC; the sides AB, AD are equal, and the side BC is less than either of them. Prove that the angle ABC is greater than the angle ADC.

[Produce BC to E, making BE equal to BA.]

10. If the side AB be greater than the side AC in the triangle ABC, and D be the middle point of BC; the angle DAC shall be greater than the angle DAB.

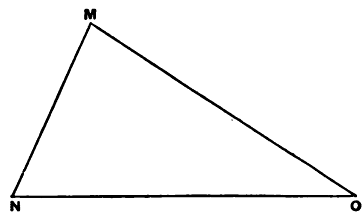
11. ABC is a triangle in which the side AB is greater than the side AC. D is the middle point of BC, and E the point in BC where the bisector of the angle A meets BC. Prove that E lies between D and C.

PROPOSITION XIX. THEOREM.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let MNO be a $\triangle$, of which the $\angle MNO$ is greater than the $\angle MON$.

The side MO is also greater than the side MN.



For if not, MO must either $= MN$, or be less than MN.

If MO were $= MN$,

the $\angle MNO$ would $=$ the $\angle MON$; I. v.

but it is not; *Hyp.*

$\therefore MO$ is not $= MN$.

Again, if MO were less than MN,

the $\angle MNO$ would be less than the $\angle MON$; I. XVIII.

but it is not; *Hyp.*

$\therefore MO$ is not less than MN.

And it has been shewn that MO is not $= MN$.

$\therefore MO$ is greater than MN.

Wherefore *the greater angle &c.* Q. E. D.

REFERENCES.

I. v. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

I. XVIII. The greater side of every triangle has the greater angle opposite to it.

EXERCISES.

1. D is any point in BC, the base of an isosceles triangle ABC; prove that AD is less than either of the equal sides.

2. BC, a side of an equilateral triangle ABC, is produced to any point D. Join AD, and shew that AD is greater than BC.

3. In the base OQ of a triangle NOQ, which has the side NO not greater than the side NQ, take any point D, and prove that ND is less than NQ.

4. If AB be greater than AC in the triangle ABC, and the bisectors of the angles B and C meet at D, BD shall be greater than CD. Also prove the converse theorem.

5. If, on the same base, and on the same side of it, there be two triangles having their sides which are terminated at one extremity of the base equal to one another, their sides terminated at the other extremity of the base shall be unequal.

6. HDF is a triangle, and the angle H is bisected by a straight line which meets DF at E; shew that HD is greater than DE and HF greater than EF.

7. In the figure to Prop. XVI., if KL be the greatest side of the triangle KLM, LP shall be the greatest side of the triangle PLM.

8. $OPQR$ is a quadrilateral of which PR is a diagonal. If OP be equal to QR, and the angles OPR, QPR be respectively greater than the angles ORP, QRP, then OR shall be the longest and PQ the shortest side of the figure.

9. If a straight line be drawn through H, one of the angular points of a square, cutting one of the opposite sides, and meeting the other produced at K; shew that HK is greater than the diagonal of the square.

10. $ABCD$ is a rhombus; from A draw AE cutting the side BC in any point, and meeting DC produced in E. Prove that AE is greater than the diagonal AC.

11. O is the middle point of the side LM in the triangle KLM; if the angle OKM be greater than the angle OKL, KL shall be greater than KM.

12. ABC is a triangle in which BA is greater than CA; the angle A is bisected by a straight line which meets BC at D. Prove that BD is greater than CD.

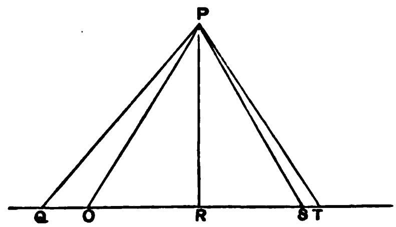
[From AB cut off AE equal to AC, and join DE.]

13. *Of all the straight lines which can be drawn from a given point to meet a given straight line the least is the perpendicular; and of the rest, that which is nearer to the perpendicular is always less than one more remote; and from the same point there can be drawn to meet the line two straight lines, and only two, which are equal to one another, one on each side of the perpendicular.*

Let QT be the given st. line, and P the given point without it.

Draw PR ⊥r to QT, and PO, PQ, any other st. lines from P meeting QT in O and Q; and let PO be nearer to PR than PQ.

Then shall PR be less than PO, and PO less than PQ.



For, ∵ the $\angle$ PRO is a rt. $\angle$,
∴ the $\angle$ POR is less than a rt. $\angle$; I. XVII.
∴ the $\angle$ PRO is greater than the $\angle$ POR;
∴ PO is greater than PR, I. XIX.
that is, PR, the perpendicular, is less than PO, any other line drawn from P to meet QT.

Again, ∵ the exterior $\angle$ POQ of the $\triangle$ POR is greater than the interior, opposite $\angle$ PRO, I. XVI.
∴ the $\angle$ POQ is an obtuse $\angle$;
∴ the $\angle$ PQO is an acute $\angle$; I. XVII.
∴ the $\angle$ POQ is greater than the $\angle$ PQO;
∴ PQ is greater than PO, I. XIX.
that is, PO, the nearer to the perpendicular, is less than PQ, the more remote.

Also there can be drawn from P to meet QT two equal st. lines, one on each side of PR.

From RT cut off RS equal to RO, and join PS.
Then ∵ $RS = RO$, *Constr.*
and RP is common to the two $\triangle$s SRP, ORP,
the 2 sides SR, RP = the 2 sides OR, RP, each to each,
and the rt. $\angle$ PRS = the rt. $\angle$ PRO; Ax. 11.
∴ the base PS = the base PO. I. IV.
But, besides PS, no other st. line equal to PO can be drawn from P to meet QT.

For, if possible, let $PT = PO$.
Then, ∵ $PT = PO$,
and $PS = PO$;
∴ $PS = PT$, Ax. 1.
that is, a line nearer to the perpendicular = a line more remote,
which has been shewn to be impossible.

Wherefore *of all the straight lines &c.* Q.E.D.

REFERENCES.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 11. All right angles are equal to one another.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. XVI. If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

I. XVII. Any two angles of a triangle are together less than two right angles.

DEFINITIONS.

The *distance from a point to a line* is the length of the perpendicular let fall from the point on the line.

The *altitude of a triangle* is the length of the perpendicular from the vertex to the base, or the base produced.

The *altitude of a parallelogram* is the length of the perpendicular on the *base* from any point in the opposite side.

EXERCISES.

14. A diagonal is greater than a side of a square.

15. A diagonal of a rectangle is greater than the longest side.

16. A side of a rhombus is greater than the half of the longer diagonal.

17. Prove that Point however deep, if standing square, is nearer to the Batsman than to the Bowler.

18. The perpendicular AD divides the base BC of a triangle ABC into two segments of which BD is the greater; prove that AB is greater than AC, and the angle BAD than the angle CAD.

19. Prove indirectly the converse of the first part of Ex. 18.

20. If two right-angled triangles have their hypotenuses equal and a side of one equal to a side of the other, prove that they are equal in all respects.

21. Two right-angled triangles which have two sides about an acute angle of one equal to two sides about an acute angle of the other, each to each, are equal in all respects.

22. If a point be taken in a straight line drawn at right angles to the diameter of a circle from the extremity of it, and this point be joined to the centre, the joining line shall cut the circumference.

23. If from the vertical angle of a triangle three straight lines be drawn, one bisecting the angle, another bisecting the base, and the third perpendicular to the base, prove that the bisector of the angle is intermediate in position and magnitude to the others.

[See Ex. 6, Prop. XIII., and Ex. 11, Prop. XVIII.]

PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

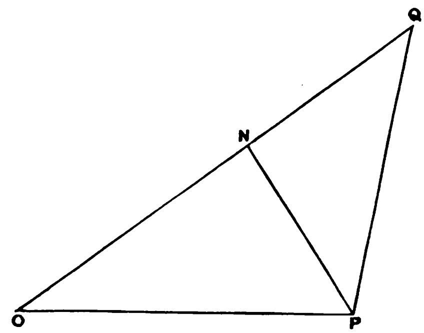
Let NOP be a $\triangle$.

Any two sides of it are together greater than the third side,

namely, ON, NP greater than PO;

NP, PO greater than ON;

and PO, ON greater than NP.



Produce ON to Q, making $NQ = NP$, { *Post.* 2. I. III.

and join QP. *Post.* 1.

Then $\because NQ = NP$, *Constr.*

the $\angle NQP =$ the $\angle NPQ$. I. v.

But the $\angle OPQ$ is greater than the $\angle NPQ$; *Ax.* 9.

$\therefore$ the $\angle OPQ$ is greater than the $\angle OQP$.

And $\because$ the $\angle OPQ$ of the $\triangle OPQ$ is greater than its $\angle OQP$,

and that the greater $\angle$ is subtended by the greater side; I. XIX.

$\therefore$ the side OQ is greater than the side PO.

But $OQ = ON$ and NP.

$\therefore$ ON, NP are greater than PO.

In the same manner it may be shewn that

NP, PO are greater than ON,

and PO, ON greater than NP.

Wherefore *any two sides &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

Ax. 9. The whole is greater than its part.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. V. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

EXERCISES.

1. In a triangle any side is greater than the difference between the other two sides.

2. The diameter is greater than any other straight line that can be drawn with its ends in the circumference of a given circle.

3. The sum of the diagonals of any quadrilateral is greater than the sum of either pair of opposite sides of the figure ; and the perimeter of a four-sided figure is less than twice the sum of the diagonals.

4. If three points be taken in the sides of a triangle, one in each, the perimeter of the triangle is greater than that of the triangle formed by joining the three points.

5. Any three sides of a quadrilateral are greater than the fourth side.

6. In a quadrilateral the sum of any two sides is greater than the difference between the other two.

7. Any side of a polygon is less than the sum of the other sides.

8. The perimeter of a triangle is greater than the double of any one side, and less than double the sum of any two sides.

9. The sum of the distances of any point from the three angles of a triangle is greater than the semi-perimeter of the triangle.

10. The four sides of any quadrilateral are greater than the two diagonals.

11. The two sides of a triangle are greater than twice the straight line drawn from the vertex to the middle point of the base.

12. Any two sides of a triangle are together less than the sum of the third side and twice the median.

13. The sum of the three medians of a triangle is less than its perimeter.

14. The sum of the distances of any point from the angles of a quadrilateral is not less than the sum of the diagonals, and greater than the semi-perimeter of the figure.

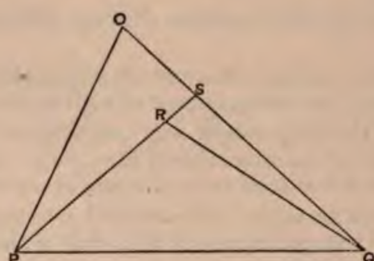
15. N is any point in PR one of the equal sides of the isosceles triangle PQR, and M is the middle point of QR; shew that the sum of PQ and MN is greater than the sum of PN and QM.

PROPOSITION XXI. THEOREM.

If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let OPQ be a $\triangle$, and from the points P, Q, the ends of the side PQ, let the two st. lines PR, QR be drawn to the point R within the $\triangle$.

Then PR, QR shall be less than PO, OQ, the other 2 sides of the $\triangle$ but shall contain an $\angle PRQ$ greater than the $\angle POQ$.



Produce PR to meet OQ at S. *Post.* 2.

$\because$ two sides of a $\triangle$ are greater than the third side,
the 2 sides PO, OS of the $\triangle$ POS are greater than the side PS. I. XX.

To each of these add SQ.

$\therefore PO$, OQ are greater than PS, SQ.

Again, the 2 sides QS, SR of the $\triangle QSR$ are greater than the side QR. I. XX.

To each of these add RP.

$\therefore QS$, SP are greater than QR, RP.

But it has been shewn that PO, OQ are greater than QS, SP;
much more then are PO, OQ greater than PR, RQ.

Again, $\because$ the extr. $\angle$ of any $\triangle$ is greater than the intr. opposite $\angle$,
the extr. $\angle PRQ$ of the $\triangle QRS$ is greater than the $\angle QSR$. I. XVI.

For the same reason,
the extr. $\angle QSP$ of the $\triangle OPS$ is greater than the $\angle POS$.

And it has been shewn that the $\angle PRQ$ is greater than the $\angle QSP$;
much more then is the $\angle PRQ$ greater than the $\angle POQ$.

Wherefore *if from the ends &c.* Q. E. D.

REFERENCES.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

I. XVI. If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

I. XX. Any two sides of a triangle are together greater than the third side.

EXERCISES.

1. A and B are two points on the same side of the straight line EC; AC is drawn perpendicular to EC and produced to D so that CD is equal to AC; BD is joined cutting EC at E, and F is any other point in EC. Prove that BE, EA are together less than BF, FA.

2. A race is arranged to be run from the clump to the wall and thence to the elm-tree. Shew which is the best course to take.

3. If two triangles ABC, ABD be on the same base AB, and the point C be not coincident with D nor without the triangle ABD, the perimeter of the triangle ACB shall be less than that of the triangle ABD.

4. ABC is a triangle, and P any point within it; shew that the sum of PA, PB, PC is less than the perimeter of the triangle.

5. If lines be drawn from the angles of a square to a point within it, they shall be together less than the perimeter of the square.

6. The perimeter of a rhombus is greater than the sum of the distances of any point within it from the angular points.

7. HKL is a triangle, and $KMNL$ a quadrilateral having the angular points M, N within the triangle HKL. Prove that the perimeter of the quadrilateral is less than that of the triangle.

8. Two convex polygons $ABFGH$, $ABCDE$ are on the same base AB and on the same side of it, none of the points F, G, H being without the polygon $ABCDE$; prove that the perimeter of $ABFGH$ is less than that of $ABCDE$.

9. If one convex polygon be within another, its perimeter shall be less than that of the other.

10. ABC, ABD are two triangles on the same base AB, and AC is equal to AD. If points P, Q be taken in AC, AD respectively, such that the angles PBC, PCB are equal, and likewise QBD and QDB equal, shew that Q cannot lie within the triangle PAB.

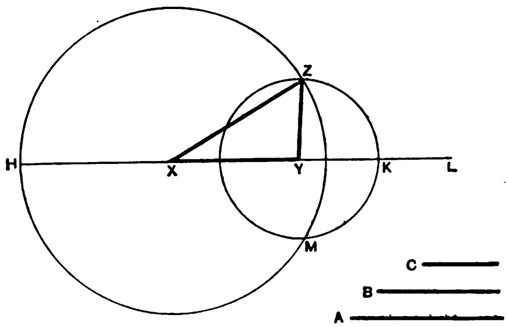
PROPOSITION XXII. PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Let A, B, C be the 3 given st. lines, of which any two whatever are greater than the third;

namely, A and B greater than C;
B and C " A;
C and A " B.

It is required to make a $\triangle$ of which the sides shall $=$ A, B, C, each to each.



Take a st. line HL terminated at H, but unlimited towards L, and make $HX = A$, $XY = B$, and $YK = C$. I. III.
From the centre X, at the distance XH, describe the circle HZM. *Post.* 3.
From the centre Y, at the distance YK, describe the circle KZM, cutting the former circle at Z, *Post.* 3.
and join ZX, ZY. *Post.* 1.

The $\triangle XYZ$ shall have its sides equal to the 3 st. lines A, B, C.

$\because$ the point X is the centre of the circle HZM,
$XH = XZ$. *Def.* 15.
But $XH = A$; *Constr.*
$\therefore XZ = A$. *Ax.* 1.

Again, $\because$ the point Y is the centre of the circle KZM,
$YK = YZ$. *Def.* 15.
But $YK = C$; *Constr.*
$\therefore YZ = C$. *Ax.* 1.
And $XY = B$. *Constr.*

$\therefore$ the 3 st. lines ZX, XY, YZ respectively $=$ the 3 st. lines A, B, C.

Wherefore *the* $\triangle XYZ$ *has its* 3 *sides* ZX, XY, YZ *equal to the* 3 *given st. lines* A, B, C. Q. E. F.

REFERENCES.

Def. 15. A circle is a plane figure contained by one line, which is called the circumference, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Post. 3. Let it be granted that a circle may be described from any centre, at any distance from that centre.

Ax. 1. Things which are equal to the same thing are equal to one another.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

EXERCISES.

1. Draw figures shewing how the construction fails when B and C together are not greater than A.

2. Find a point at given distances from two given points.

3. Is it possible to make a triangle having its sides respectively 4, 5, and 10 inches in length?

4. Construct a triangle having its sides respectively twice, three times, and four times a given line.

5. Construct a triangle of which each side is equal to half the sum of the other two sides.

6. Construct a triangle having given the base, and also the sum and difference of the two sides.

7. Given the sums of each pair of sides, construct the triangle.

8. Construct a triangle, having given two sides, and the line from the middle point of one of them to the opposite angle.

9. Construct a quadrilateral $PQRS$, having given SP, PQ, QS, SR, RQ.

10. Construct a quadrilateral, having given that the four sides are respectively equal to four given straight lines, and that one of the angles, contained by two specified sides, is a right angle.

11. Construct a pentagon, having given that the sides and two diagonals are respectively equal to seven given straight lines.

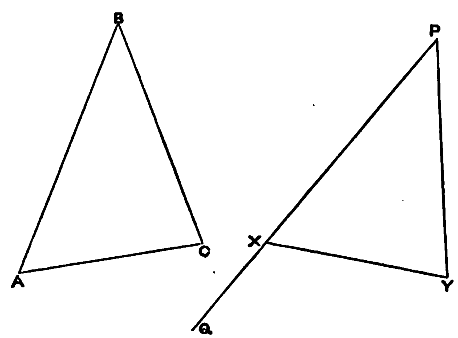
12. From a given point draw three lines respectively equal to three given lines A, B, and C, of which the sum of A and B is greater than the double of C, so that their extremities may be in one line and equally distant from each other.

PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let PQ be the given st. line, P the given point in it, and ABC the given rectilineal $\angle$.

It is required to make at the given point P, in the given st. line PQ, an $\angle$ equal to the given rectilineal $\angle ABC$.



In BA, BC take any points A, C, and join AC. *Post.* 1.

Make the $\triangle PXY$, the sides of which shall $=$ the 3 st. lines, BA, AC, CB;

so that PX shall $= BA$, $XY = AC$, and $YP = CB$. I. XXII.

Then the $\angle XPY$ shall $=$ the $\angle ABC$.

$\because XP, PY = AB, BC$, each to each,

and the base $XY =$ the base AC; *Constr.*

$\therefore$ the $\angle XPY =$ the $\angle ABC$. I. VIII.

Wherefore *at the given point P in the given st. line PQ, the $\angle XPY$ has been made equal to the given rectilineal $\angle ABC$.* Q. E. F.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

I. VIII. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

I. XXII. To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

EXERCISES.

1. At a given point in a given straight line there cannot be made more than two angles equal to a given angle, one on each side of the line.

2. At a given point in a given straight line make an angle supplementary to a given angle.

3. At a given point in a given straight line make an angle which is equal to the complement of a given angle.

4. Make an angle equal to three times a given angle, and such that the two angles have a common vertex and a common arm.

5. Make an angle $3\frac{1}{2}$ times a given angle. Is it possible with Euclid's restrictions to make an angle $2\frac{1}{3}$ times another?

6. The sum and difference of two angles are together double of the greater of them.

7. Find two angles whose sum is equal to a given angle and whose difference is equal to another given angle.

8. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

9. A is a given point without a given straight line, and B a given point in it. Draw from A to the given straight line a line AP, such that the sum of AP and PB may be equal to a given line.

10. P is a given point without a given straight line, and O a given point in it. Find a point X in the given line, such that the difference between the lines PX and OX may be equal to a given line.

11. Construct a triangle, having given:

    (i) Two sides and the included angle.

    (ii) A side and the adjacent angles.

    (iii) The base, an angle at the base, and the sum of the sides.

    (iv) The base, an angle at the base, and the difference of the sides.

12. Construct a rhombus, given a side and an angle.

13. Construct a polygon equiangular to a given polygon.

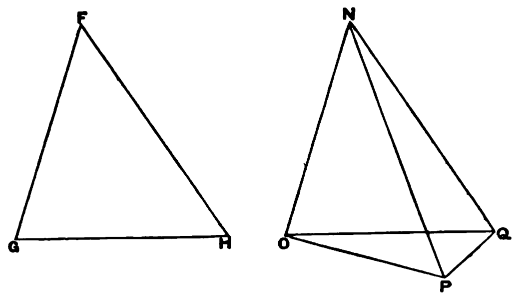
14. From the angular points E, F of the triangle DEF draw EG, FH, making equal angles with EF, meeting the opposite sides in G and H, and intersecting in K. Prove that, if the angle DKH is equal to the angle DKG, the triangle is isosceles.

PROPOSITION XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let *FGH, NOP* be 2 $\triangle$s, which have the two sides *FG, FH* equal to the two sides *NO, NP*, each to each, namely, *FG* to *NO*, and *FH* to *NP*, but the $\angle\, GFH$ greater than the $\angle\, ONP$.

Then the base *GH* shall be greater than the base *OP*.



Of the two sides *NO, NP*, let *NO* be the side not greater than the other.

At the point *N* in the st. line *NO*, make the $\angle\, ONQ$ equal to the $\angle\, GFH$; I. XXIII.

make *NQ* equal to *FH* or *NP*, I. III.

and join *OQ, QP*. *Post.* 1.

$\because FG = NO$, *Hyp.*

and $FH = NQ$; *Constr.*

the 2 sides *GF, FH* = the 2 sides *ON, NQ*, each to each,

and the $\angle\, GFH$ = the $\angle\, ONQ$; *Constr.*

$\therefore$ the base *GH* = the base *OQ*. I. IV.

And $\because NQ = NP$, *Constr.*

the $\angle\, NQP$ = the $\angle\, NPQ$. I. V.

But the $\angle\, NPQ$ is greater than the $\angle\, OQP$. *Ax.* 9.

$\therefore$ the $\angle\, NPQ$ is greater than the $\angle\, OQP$.

Much more then is the $\angle\, OPQ$ greater than the $\angle\, OQP$. *Ax.* 9.

And $\because$ in the $\triangle\, OPQ$, the $\angle\, OPQ$ is greater than the $\angle\, OQP$,

and that the greater $\angle$ is subtended by the greater side;

$\therefore$ the side *OQ* is greater than the side *OP*. I. XIX.

But *OQ* was shewn to = *GH*;

$\therefore$ *GH* is greater than *OP*.

Wherefore *if two triangles &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 9. The whole is greater than its part.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. V. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

I. XIX. The greater angle of every triangle is subtended by the greater side.

I. XXIII. At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

EXERCISES.

1. In the figure, prove that P must fall below OQ.

[See Ex. 3, Prop. XIX.]

2. Shew that in using compasses, if we increase the angle between the legs, we get a longer radius.

3. ABC is an acute-angled triangle of which AC is the greatest side. Produce CB to D making BD equal to BC and join AD. Shew that AD is greater than AC.

4. The vertical angle BAC of the isosceles triangle ABC is bisected by the straight line AD. Shew that the distance of any point P from B is less than its distance from C, if it is on the same side of AD as B is.

5. From F, the middle point of DE, FG is drawn at right angles to DE; shew that every point out of the line FG is at unequal distances from D and E.

6. From E, the middle point of CD, EF is drawn making an angle which is five-elevenths of two right angles with ED. Shew that, if F be any point in EF, F is nearer to D than to C.

7. The greater angle at the centre of a circle is subtended by the greater chord.

8. If the hypotenuse BC of a right-angled triangle be produced to D so that CD is equal to AB, AD shall be greater than BC.

9. If a point O be taken within an equilateral triangle ABC so that the angle OAB is greater than the angle OAC, the angle OCB shall be greater than the angle OBC.

10. Of two sides of a triangle, that is the greater which is cut by the perpendicular bisector of the base.

11. If one side of a triangle be greater than another, the perpendicular on it from the opposite angle shall be less than the corresponding perpendicular on the other side.

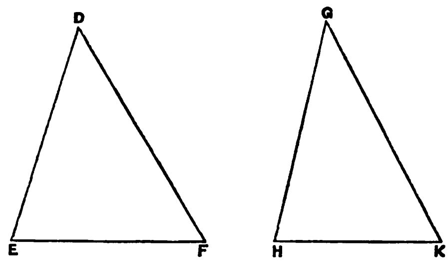
[Produce each perpendicular its own length, and join to the angle from which the other perpendicular is drawn.]

PROPOSITION XXV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them, of the other.

Let DEF, GHK be two $\triangle$s, which have the two sides DE, DF equal to the two sides GH, GK, each to each, namely, DE to GH, and DF to GK, but the base EF greater than the base HK.

The $\angle EDF$ shall be greater than the $\angle HGK$.



For if not, the $\angle EDF$ must either = the $\angle HGK$, or be less than it.

If the $\angle EDF$ were = the $\angle HGK$,

the base EF would = the base HK; I. IV.

but it is not; *Hyp.*

$\therefore$ the $\angle EDF$ is not = the $\angle HGK$.

Again, if the $\angle EDF$ were less than the $\angle HGK$,

the base EF would be less than the base HK; I. XXIV.

but it is not; *Hyp.*

$\therefore$ the $\angle EDF$ is not less than the $\angle HGK$.

And it has been shewn that the $\angle EDF$ is not = the $\angle HGK$;

$\therefore$ the $\angle EDF$ is greater than the $\angle HGK$.

Wherefore *if two triangles &c.* Q. E. D.

REFERENCES.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. XXIV. If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.

EXERCISES.

1. Shew that when taking a longer radius with a pair of compasses we increase the angle between the legs.

2. PMN is a triangle having the side PM greater than the side PN. Draw PO to the middle point of MN and prove that POM is an obtuse angle.

3. In a circle the longer chord subtends the greater angle at the centre.

4. Four unequal rods on AB, BC, CD, DA are jointed together by hinges, thus forming a quadrilateral. Shew that, if the angle BAD is made larger, the angle BCD will also be increased.

5. A pole OP is erected on the intersection O of the diagonals of a horizontal square $ABCD$, and its top is attached to cords fastened to pegs at the corners of the square; AP is 26 ft.; BP, 30 ft.; CP, 28 ft.; DP, 30 ft.; in what direction will the pole slant from the vertical?

6. ABC is an isosceles triangle, whose vertex is A, and P is any point. If the angle PBC be greater than the angle PCB, then shall the angle PAC be greater than the angle PAB.

7. $ABCD$ is a quadrilateral having AD equal to BC, but AC greater than BD; shew that the angle ABC is greater than the angle BAD, and the angle ADC greater than BCD.

8. If perpendiculars be let fall from two angles of a triangle on the opposite sides, prove that the side on which the greater perpendicular falls is less than the side on which the less falls.

9. $MNOP$ is a quadrilateral, having MN equal to OP, but the angle PON greater than the angle MNO; shew that the angle NMP is greater than the angle OPM.

10. HGK is a triangle, having the side HG greater than the side HK, and M is the middle point of GK. Join HM, and prove that any point in HM is nearer to K than to G.

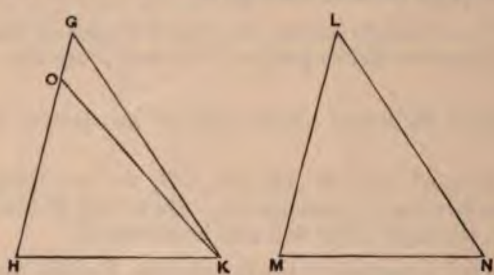
PROPOSITION XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one to the third angle of the other.

Let GHK, LMN be two $\triangle$s, which have the $\angle$s GHK, HKG equal to the $\angle$s LMN, MNL, each to each, namely, GHK to LMN, and HKG to MNL; and also one side equal to one side.

First, let those sides be equal which are adjacent to the equal $\angle$s in the two $\triangle$s, namely, HK to MN.

Then the other sides shall be equal, each to each, namely, GH to LM, and GK to LN, and the third $\angle\, HGK$ to the third $\angle\, MLN$.



For, if GH be not $= LM$, one of them must be the greater.
Let GH be the greater, and make $HO = LM$, — I. III.
and join OK. — *Post.* 1.
Then $\because OH = LM$, — *Constr.*
and $HK = MN$, — *Hyp.*
the 2 sides OH, HK = the 2 sides LM, MN, each to each,
and the $\angle\, OHK$ = the $\angle\, LMN$; — *Hyp.*
$\therefore$ the $\triangle\, OHK$ = the $\triangle\, LMN$,
and the other $\angle$s = the other $\angle$s, each to each, to which the equal sides are opposite;
$\therefore$ the $\angle\, OKH$ = the $\angle\, LNM$. — I. IV.
But the $\angle\, GKH$ = the $\angle\, LNM$. — *Hyp.*
$\therefore$ the $\angle\, OKH$ = the $\angle\, GKH$, — *Ax.* 1.
the less to the greater; which is impossible. — *Ax.* 9.
$\therefore GH$ is not unequal to LM,
that is, $GH = LM$;
and $HK = MN$; — *Hyp.*
$\therefore$ the 2 sides GH, HK = the 2 sides LM, MN, each to each;
and the $\angle\, GHK$ = the $\angle\, LMN$; — *Hyp.*
$\therefore$ the base GK = the base LN,
and the third $\angle\, HGK$ = the third $\angle\, MLN$. — I. IV.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 9. The whole is greater than its part.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

EXERCISES.

1. If a straight line bisecting the vertical angle of a triangle be at right angles to the base, the triangle is isosceles.

2. Two trees at noon cast shadows of the same length; prove that their height is the same.

3. A football ground was 100 yds. long. Starting from the 50 yds. flag I walked away from the ground at right angles to the touch line and, after going 30 yds., found that the 25 yds. flag was in a direct line with the nearer goal post. What was the distance from the nearer goal post to the corner flag?

4. A straight line bisects the angle D of the triangle DEF; from E a perpendicular is drawn to this bisecting line meeting it at G, and EG is produced to meet DF or DF produced at H; shew that EG is equal to GH.

5. Through a given point within an angle BAC draw a straight line cutting off equal parts from AB, AC.

6. OM, ON are two straight lines meeting at O, and P is any point. Through P draw a straight line equally inclined to OM, ON.

7. PQR is an isosceles triangle having PQ equal to PR; the angles PQR, PRQ are bisected by QS, RT, meeting the opposite sides in S, T; prove that QS is equal to RT, and PT to PS.

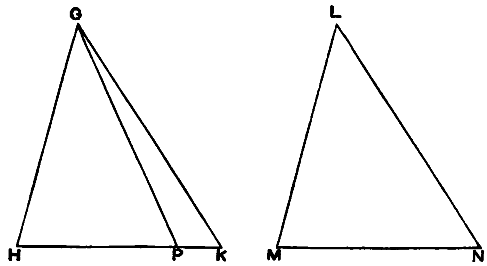
8. On sailing from a pier I notice the fort F to be due S. and another fort G to be exactly N.W. I steer W.S.W. and after a time find the fort F to lie E. from me, and observe the fort G in the N.E. Prove that the forts are equally distant from the pier.

9. Walking at a uniform rate along the beach in the direction from E. to W., I notice at 9.5 a black buoy due S. of me; at 9.13 I see a red buoy to the S.W.; at 9.30 turning round I observe the black buoy to be in the S.E.; and at 9.38 the red buoy is directly opposite to me in the S. Prove that the buoys are equally distant from the beach.

PROPOSITION XXVI. (*Continued.*)

Next, let sides which are opposite to equal $\angle$s in each $\triangle$ be equal to one another, namely, GH to LM.

Then in this case likewise the other sides shall be equal, each to each, namely, HK to MN, and GK to LN, and the third $\angle HGK$ to the third $\angle MLN$.



For, if HK be not $= MN$, one of them must be the greater.
Let HK be the greater, and make $HP = MN$, — I. III.
and join GP. — *Post.* 1.
Then $\because GH = LM$, — *Hyp.*
and $HP = MN$, — *Constr.*
the 2 sides GH, HP = the 2 sides LM, MN, each to each,
and the $\angle GHP$ = the $\angle LMN$; — *Hyp.*
$\therefore$ the $\triangle GHP$ = the $\triangle LMN$,
and the other $\angle$s = the other $\angle$s, each to each, to which the equal sides are opposite;
$\therefore$ the $\angle HPG$ = the $\angle MNL$. — I. IV.
But the $\angle HKG$ = the $\angle MNL$. — *Hyp.*
$\therefore$ the $\angle HPG$ = the $\angle HKG$; — *Ax.* 1.
that is, the extr. $\angle HPG$ of the $\triangle GPK$ = its intr. opposite $\angle HKG$;
which is impossible. — I. XVI.
$\therefore$ HK is not unequal to MN,
that is, $HK = MN$;
and $GH = LM$; — *Hyp.*
$\therefore$ the 2 sides GH, HK = the 2 sides LM, MN, each to each;
and the $\angle GHK$ = the $\angle LMN$; — *Hyp.*
$\therefore$ the base GK = the base LN,
and the third $\angle HGK$ = the third $\angle MLN$. — I. IV.

Wherefore *if two triangles &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 1. Things which are equal to the same thing are equal to one another.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. XVI. If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

EXERCISES.

1. If two right-angled triangles have equal hypotenuses, and an acute angle of one equal to an acute angle of the other, the triangles are equal in all respects.

2. The perpendicular on the base of an isosceles triangle from the vertex bisects the base and the vertical angle.

3. Shew that a circle is symmetrical with regard to a diameter.

4. Take as centre a point P without a straight line AB, and describe two circles cutting AB. Prove that the parts of AB intercepted between the circles will be equal.

5. From the top of a tower two objects on the ground have equal angles of depression; shew that they are equally distant.

6. The perpendiculars let fall on the opposite sides from the extremities of the base of an isosceles triangle are equal.

7. Let ABC be a triangle, and let the angles CAB, ABC be bisected by the straight lines AD, BD, meeting at D. From D draw DE, DF, DG perpendiculars to AB, BC, CA. Prove that DE, DF, DG are equal to one another.

8. In BC one of the sides of a triangle ABC find a point equidistant from AB, AC.

9. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines which intersect shall be equal.

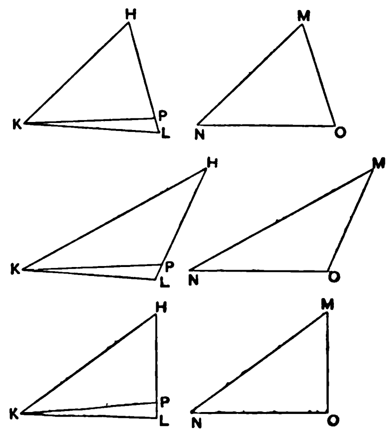
10. Find a point equally distant from three given straight lines.

11. Through an angle of a given triangle draw a straight line cutting the opposite side such that perpendiculars let fall on it from the other two angles may be equal.

12. Through a given point draw a line such that perpendiculars on it from two other given points may be on opposite sides of it and equal to each other.

13. *If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to one pair of equal sides equal; then, if the angles opposite to the other pair of equal sides be both acute, or both obtuse angles, or if one of them be a right angle, the two triangles are equal in all respects.*

Let *HKL*, *MNO* be two △s having the two sides *HK*, *KL* equal to the two sides *MN*, *NO*, each to each, and the ∠*KHL* equal to the ∠*NMO*.
Then, if the ∠s *KLH*, *NOM* be both acute, or both obtuse,
or if one of them be a rt. ∠,
the △ *HKL* shall = the △ *MNO* in all respects.



If the ∠ *HKL* be not = the ∠ *MNO*,
one of them must be greater than the other.
Let the ∠ *HKL* be the greater.
Make the ∠ *HKP* = the ∠ *MNO*, and let *KP* meet *HL* in *P*. I. XXIII.
Then the △s *HKP*, *MNO* are equal in all respects. I. XXVI.
∴ the ∠ *KPH* = the ∠ *NOM*,
and *KP* = *NO*;
but *KL* = *NO*; *Hyp.*
∴ *KP* = *KL*; *Ax.* 1.
∴ the ∠ *KPL* = the ∠ *KLP*; I. V.
and it has been shewn that the ∠ *KPH* = the ∠ *NOM*;
∴ the ∠s *KPL*, *KPH* = the ∠s *KLP*, *NOM*. *Ax.* 2.
But the ∠s *KPL*, *KPH* together = two rt. ∠s. I. XIII.
∴ the ∠s *KLH*, *NOM* together = two rt. ∠s.

Now, let the ∠s *KLH*, *NOM* be both acute;
then two acute ∠s = two rt. ∠s; which is impossible.
Next, let the ∠s *KLH*, *NOM* be both obtuse;
then two obtuse ∠s = two rt. ∠s; which is impossible.
Lastly, let one of the ∠s *KLH*, *NOM* be a rt. ∠;
then the other is also a rt. ∠.
But the ∠ *NOM* has been shewn = the ∠ *KPH*;
∴ the ∠ *KPH* = the ∠ *KLH*,
the extr. ∠ of the △ *KPL* = the intr. opposite ∠; which is impossible. I. XVI.

∴ the ∠ *HKL* is not unequal to the ∠ *MNO*.
∴ the ∠ *HKL* = the ∠ *MNO*;
and ∴ the △ *HKL* = the △ *MNO* in all respects. I. IV.
Wherefore *if two triangles &c.* Q. E. D.

The $\angle$ s KLH, NOM will be both acute

(a) When the equal $\angle$s KHL, NMO are obtuse, or are rt. $\angle$s ; or

(b) When HK is less than KL, and consequently MN less than NO.

[See Ex. 2, Prop. XVII.; and Ex. 2, Prop. XVIII.]

REFERENCES.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 2. If equals be added to equals the wholes are equal.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. V. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

I. XIII. The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

I. XVI. If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

I. XXIII. At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

EQUALITY OF TRIANGLES.

A triangle has six parts, three sides and three angles, and in order that two triangles may be proved equal in all respects not less than three parts of one must be given equal to corresponding parts of the other. In Euclid's elements there are Propositions IV., VIII., and XXVI., by which triangles may be proved identically equal. The parts given equal are :

in Prop. IV., two sides and the contained angle;
" VIII., the three sides;
" XXVI., two angles and the adjacent side; and
two angles and a side opposite to one of them.

Thus there remain the cases in which the following parts in one triangle are given equal to corresponding parts in the other :

(1) two sides and an angle opposite to one of them;

(2) the three angles.

When two triangles have two sides of the one respectively equal to two sides of the other, and the angles opposite to one pair of equal sides equal, it does not always follow that the triangles are equal, but with certain restrictions as to the magnitude of the angles opposite to the other pair of equal sides, it has been proved on the opposite page that the triangles are equal in all respects.

It will be shewn in Prop. XXXII. that the three angles of a triangle are not independent of each other, that is to say, when two of them are given, the third can be found, so that when the three angles of one triangle are given equal respectively to the three angles of another, only two parts are really given equal to two parts, and the triangles will not necessarily be equal in every respect. It will follow from Prop. XXIX. that triangles, which have their sides respectively parallel to those of another triangle, will be equiangular to one another, but may all be unequal.

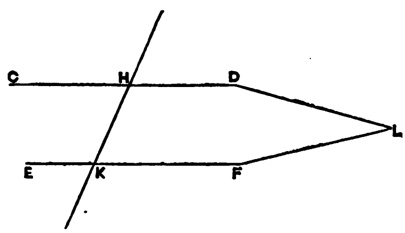
SECTION II.

PROPOSITION XXVII. THEOREM.

If a straight line falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

Let the st. line *HK*, which falls on the two st. lines *CD, EF*, make the alternate $\angle$s *CHK, HKF* equal to one another.

CD shall be $\parallel$ to *EF*.



For if not, *CD* and *EF*, being produced, will meet, either towards *D, F*, or towards *C, E*.

Let them be produced and meet towards *D, F* at the point *L*. *Post.* 2.

Then *LHK* is a $\triangle$, and its extr. $\angle$ *CHK* is greater than the intr. opposite $\angle$ *HKL*. I. XVI.

But the $\angle CHK =$ the $\angle HKL$; *Hyp.*

which is impossible.

$\therefore$ *CD* and *EF* being produced do not meet towards *D, F*.

In like manner, it may be shewn that they do not meet towards *C, E*.

But those st. lines in the same plane, which being produced ever so far both ways do not meet, are parallel. *Def.* 35.

$\therefore$ *CD* is $\parallel$ to *EF*.

Wherefore *if a straight line &c.* Q. E. D.

REFERENCES.

Def. 35. Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

I. xvi. If a side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

EXERCISES.

1. If a straight line falling on two other straight lines make the alternate angles unequal, the two straight lines shall not be parallel.

2. A square has its opposite sides parallel.

3. A rhombus is a parallelogram.

4. If two opposite sides of a quadrilateral be equal to one another, and the remaining sides be also equal to one another, the figure is a parallelogram.

5. If straight lines be perpendicular to the same straight line, they shall be parallel to one another.

6. A straight line is drawn through the vertex of an isosceles triangle so as to make equal angles with the sides; prove that this line must be either parallel to the base, or perpendicular to it.

7. If a straight line falling on two other straight lines make the exterior angles upon opposite sides of the line equal to one another, the two straight lines shall be parallel.

8. If a straight line falling on two other straight lines make two exterior angles upon the same side of the line together equal to two right angles, the two straight lines shall be parallel.

9. If two straight lines bisect each other, and their extremities be joined, the figure thus formed shall be a parallelogram.

10. If the diagonals of a quadrilateral bisect each other, it is a parallelogram.

11. A diameter is drawn in each of two concentric circles. Prove that the straight lines joining their extremities are the sides of a parallelogram.

12. The side AC of a triangle ABC is bisected at D, and BD is produced to E, so that DE is equal to BD. Prove that AE is parallel to BC.

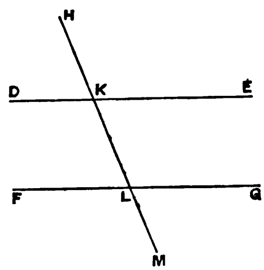
13. Two straight lines drawn from the extremities of the base of a triangle to the opposite sides cannot bisect each other.

PROPOSITION XXVIII. THEOREM.

If a straight line falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the st. line HM, which falls on the two st. lines DE, FG, make the extr. $\angle HKE$ equal to the intr. and opposite $\angle KLG$ on the same side,
or make the intr. $\angle$s on the same side EKL, KLG together equal to two rt. $\angle$s.

DE shall be $\parallel$ to FG.



$\because$ the $\angle HKE =$ the $\angle KLG$, *Hyp.*
and the $\angle HKE =$ the $\angle DKL$; I. XV.
$\therefore$ the $\angle DKL =$ the $\angle KLG$; *Ax.* 1.
and they are alternate $\angle$s;
$\therefore DE$ is $\parallel$ to FG. I. XXVII.

Again, $\because$ the $\angle$s EKL, KLG together $= 2$ rt. $\angle$s, *Hyp.*
and the $\angle$s EKL, DKL together $= 2$ rt. $\angle$s; I. XIII.
$\therefore$ the $\angle$s EKL, $DKL =$ the $\angle$s EKL, KLG. { *Ax.* 11. *Ax.* 1.
Take away the common $\angle EKL$.
$\therefore$ the remaining $\angle DKL =$ the remaining $\angle KLG$; *Ax.* 3.
and they are alternate $\angle$s;
$\therefore DE$ is $\parallel$ to FG. I. XXVII.

Wherefore *if a straight line &c.* Q. E. D.

REFERENCES.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 3. If equals be taken from equals the remainders are equal.

Ax. 11. All right angles are equal to one another.

I. XIII. The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

I. XV. If two straight lines cut one another, the vertical, opposite angles shall be equal.

I. XXVII. If a straight line falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

EXERCISES.

1. If a straight line falling on two other straight lines, make the exterior angle unequal to the interior and opposite angle on the same side of the line, the two straight lines shall not be parallel.

2. A square is a parallelogram.

3. Straight lines which are perpendicular to the same straight line are parallel to one another.

4. Two opposite angles of a quadrilateral are together equal to two right angles, and one of them is equal to a third angle of the figure; prove that the quadrilateral has two sides parallel.

5. A quadrilateral $ABCD$ has the angles ABC, BCD supplementary, as also the angles ABC, BAD. Prove that $ABCD$ is a parallelogram.

6. Two triangles ABC, DEF, having the angle ABC equal to the angle DEF, and the angle ACB to the angle DFE, are placed so that BC and EF are in the same straight line $BCEF$, and the triangles on the same side of it. Prove that neither BA, FD are parallel nor AC, DE; but that BA, AC, FD, DE, when produced, will form the sides of a parallelogram.

7. If one of the equal sides of an isosceles triangle be produced through the vertex, the bisector of the exterior angle shall be parallel to the base of the triangle.

8. If a straight line falling on two other straight lines make two exterior angles upon the same side of the line together equal to two right angles, the two straight lines shall be parallel.

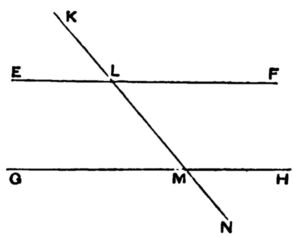
9. If a straight line falling on two other straight lines make the exterior angles upon opposite sides of the line equal to one another, the two straight lines shall be parallel.

PROPOSITION XXIX. THEOREM.

If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

Let the st. line *KLMN* fall on the two parallel st. lines *EF, GH.*

The alternate $\angle$s *ELM*, *LMH* shall = one another;
the extr. $\angle$ *KLF* shall = the intr. and opposite $\angle$ *LMH* on the same side of *KN*,
and the two intr. $\angle$s *FLM, LMH* on the same side of *KN* shall together = 2 rt. $\angle$s.



For, if the $\angle$ *ELM* be not = the $\angle$ *LMH*,
one of them must be greater than the other.
Let the $\angle$ *ELM* be the greater.
Then the $\angle$ *ELM* is greater than the $\angle$ *LMH*.
To each add the $\angle$ *FLM*.
$\therefore$ the $\angle$s *ELM, FLM* are greater than the $\angle$s *FLM, LMH*. *Ax.* 4.
but the $\angle$s *ELM, FLM* together = 2 rt. $\angle$s. I. XIII.
$\therefore$ the $\angle$s *FLM, LMH* are together less than 2 rt. $\angle$s.
But, if a st. line meet two st. lines, so as to make the two intr. $\angle$s on the same side of it, taken together, less than two rt. $\angle$s, these st. lines, being continually produced, shall at length meet on that side on which are the $\angle$s which are less than two rt. $\angle$s. *Ax.* 12.
$\therefore$ the st. lines *EF, GH*, if continually produced, will meet.
But they never meet, since they are parallel. *Hyp.*
$\therefore$ the $\angle$ *ELM* is not unequal to the $\angle$ *LMH*.
$\therefore$ the $\angle$ *ELM* = the $\angle$ *LMH*.
But the $\angle$ *ELM* = the $\angle$ *KLF*. I. XV.
$\therefore$ the $\angle$ *KLF* = the $\angle$ *LMH*. *Ax.* 1.
To each add the $\angle$ *FLM*.
$\therefore$ the $\angle$s *KLF, FLM* = the $\angle$s *FLM, LMH*. *Ax.* 2.
And the $\angle$s *KLF, FLM* together = 2 rt. $\angle$s; I. XIII.
$\therefore$ the $\angle$s *FLM, LMH* together = 2 rt. $\angle$s. *Ax.* 1.
Wherefore *if a straight line &c.* Q. E. D.

REFERENCES.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 2. If equals be added to equals the wholes are equal.

Ax. 4. If equals be added to unequals the wholes are unequal.

Ax. 12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

I. XIII. The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

I. XV. If two straight lines cut one another, the vertical, opposite angles shall be equal.

EXERCISES.

1. Through the same point there cannot be two parallels to the same line.

2. If a straight line cut one of two parallels, it will cut the other.

3. Any two adjacent angles of a parallelogram cannot be together less than two right angles.

4. The straight line which is perpendicular to one of two parallel straight lines is also perpendicular to the other.

5. If a parallelogram has two adjacent angles equal, it is a rectangle.

6. Any straight line parallel to the base of an isosceles triangle is equally inclined to the sides.

7. ST is drawn parallel to QR, one of the sides of an equilateral triangle PQR; prove that PST is an equilateral triangle.

8. The straight line drawn through the vertex of an isosceles triangle parallel to the base bisects the exterior angle at the vertex.

9. If the straight line drawn through the vertex of a triangle parallel to the base bisects the exterior angle at the vertex, the triangle is isosceles.

10. A quadrilateral $ABCD$ has the side BC parallel to AD and equal to AB; prove that AC bisects the angle BAD.

11. If two straight lines AB, CD are respectively perpendicular to two parallel straight lines EF, GH, prove that AB is parallel to CD.

12. Bisect the sides AB, AC of the triangle ABC at the points D, E, from which draw perpendiculars to AB, AC respectively; these perpendiculars shall meet.

13. Let D, E, F be the middle points of the sides BC, CA, AB of the triangle ABC; from D, E draw DG, EG at right angles to BC, CA, and meeting at G; prove that GF is perpendicular to AB. *This is usually set as follows:* The straight lines drawn at right angles to the sides of a triangle through the middle points of the sides are concurrent (i.e. meet in the same point).

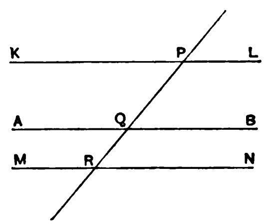
14. If OM, ON be respectively parallel to QP, QR, the angles MON, PQR are either equal or supplementary.

PROPOSITION XXX. THEOREM.

Straight lines which are parallel to the same straight line are parallel to each other.

Let KL, MN be each of them $\parallel$ to AB.

KL shall be $\parallel$ to MN.



Let the st. line PQR cut KL, AB, MN in the points P, Q, R.

Then, $\because$ KL is $\parallel$ to AB, and PQR cuts them,

the $\angle KPQ$ = the alternate $\angle PQB$. I. XXIX.

Again, $\because$ AB is $\parallel$ to MN, and PQR cuts them,

the extr. $\angle PQB$ = the intr. and opposite $\angle QRN$. I. XXIX.

And it was shewn that the $\angle PQB$ = the $\angle KPQ$;

$\therefore$ the $\angle KPR$ = the $\angle PRN$; *Ax.* 1.

and they are alternate $\angle$ s;

$\therefore$ KL is $\parallel$ to MN. I. XXVII.

Wherefore *straight lines &c.* Q. E. D.

REFERENCES.

Ax. 1. Things which are equal to the same thing are equal to one another.

I. XXVII. If a straight line falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

I. XXIX. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

EXERCISES.

1. Give the demonstration of this proposition when KL and MN are on the same side of AB.

2. Two straight lines AB, AC are each of them parallel to another straight line MN; prove that they are in one and the same straight line.

3. Two parallelograms have a common side; prove that the sides opposite to this common side are parallel to each other.

4. If two adjacent sides of a parallelogram be parallel respectively to two adjacent sides of another parallelogram, the other sides shall also be parallel, each to each.

5. MON is any angle, and at the points M, N are made outside the angle MON the two angles OMP, ONQ, such that their sum is equal to the angle MON. Prove that MP is parallel to NQ.

6. MON is any angle, and at the points M, N are made, one outside the angle MON and the other inside it, the two angles OMP, ONQ such that their difference is equal to the angle MON. Prove that MP and NQ are parallel.

7. AB and CD are parallel straight lines, P any point in AB, and Q any point in CD; PQ is joined and bisected in R. All straight lines drawn through R and terminated by the parallels shall be bisected in R.

8. Two straight lines passing through a point equidistant from two parallel straight lines shall intercept equal portions of these parallels.

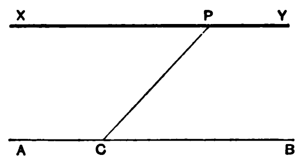
9. If two straight lines A and B are respectively parallel to two other lines C and D, the bisectors of the angles at the intersection of A and B will form with the bisectors of the angles at the intersection of C and D a right-angled parallelogram.

10. If one of the equal sides of an isosceles triangle be bisected at D and also be doubled by being produced through the extremity of the base to E, then the distance of the other extremity of the base from E is double its distance from D.

PROPOSITION XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let P be the given point, and AB the given st. line.
It is required to draw a st. line through P parallel to AB.



In AB take any point C, and join PC; *Post.* 1.
at the point P in the st. line PC, make the $\angle CPX$ equal to the $\angle PCB$, on the side of PC opposite to CB; I. XXIII.
produce the st. line XP to Y. *Post.* 2.
XY shall be $\parallel$ to AB.
$\because$ the st. line PC, meeting the two st. lines AB, XY, makes the alternate $\angle$s XPC, PCB = one another, *Constr.*
$\therefore$ XY is $\parallel$ to AB. I. XXVII.
Wherefore *the st. line XY is drawn through P, parallel to the given st. line AB.* Q. E. F.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

I. XXIII. At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

I. XXVII. If a straight line falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

EXERCISES.

1. PQR is an isosceles triangle having PQ equal to PR. Through any point S in QR draw ST meeting PR in T, so that TSR may be an isosceles triangle.

2. If ACB, ACD be adjacent angles, any parallel to BD meets the bisectors of the angles ACB, ACD at points equidistant from the point at which it intersects AC.

3. Through a given point draw a straight line making with a given straight line an angle equal to a given angle.

4. Given the altitude and the base angles of a triangle, construct it.

5. Construct a parallelogram, having given two adjacent sides and a diagonal.

6. Construct a rhombus having two opposite sides in two given parallel straight lines.

7. Through the angular points of a triangle draw parallels to the sides and shew that the triangle thus formed has its angles respectively equal to those of the given triangle.

8. OX, OY are two straight lines intersecting at O, and P is any point within the angle XOY. Draw a straight line QPR, which shall meet OX, OY at Q and R, and be bisected at P. If XOY be a right angle, and OP be 3 inches long, what is the length of QR?

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9. Construct a triangle with a given perimeter, and equiangular to a given triangle.

10. Describe an equilateral triangle of given perimeter.

11. Trisect a given finite straight line.

12. Construct a right-angled triangle, having given the perimeter and an acute angle.

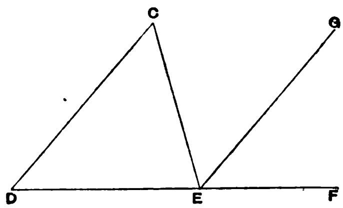
13. Construct an isosceles right-angled triangle of given perimeter.

14. MN is the hypotenuse of a right-angled triangle MON; find a point P in MN such that PN may be equal to the perpendicular from P on MO.

PROPOSITION XXXII. THEOREM.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Let CDE be a $\triangle$, and let one of its sides DE be produced to F.
The extr. $\angle\ CEF$ shall = the two intr. and opposite $\angle$s ECD, CDE;
and the 3 intr. $\angle$s DEC, ECD, CDE shall = 2 rt. $\angle$s.



Through E draw EG parallel to DC. I. XXXI.
Then $\because$ EG is $\parallel$ to DC, and CE falls on them,
the $\angle\ CEG$ = the alternate $\angle\ ECD$. I. XXIX.
Again $\because$ EG is $\parallel$ to DC, and DEF falls on them,
the extr. $\angle\ GEF$ = the intr. and opposite $\angle\ CDE$. I. XXIX.
But it was shewn that the $\angle\ CEG$ = the $\angle\ ECD$;
$\therefore$ the whole extr. $\angle\ CEF$ = the two intr. and opposite $\angle$s ECD, CDE. *Ax.* 2.

To each of these equals add the $\angle\ DEC$.
$\therefore$ the $\angle$s DEC, CEF = the 3 $\angle$s DEC, ECD, CDE. *Ax.* 2.
But the $\angle$s DEC, CEF = 2 rt. $\angle$s. I. XIII.
$\therefore$ the $\angle$s DEC, ECD, CDE = 2 rt. $\angle$s. *Ax.* 1.
Wherefore *if a side of any triangle &c.* Q. E. D.

REFERENCES.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 2. If equals be added to equals the wholes are equal.

I. XIII. The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

I. XXIX. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

I. XXXI. To draw a straight line through a given point parallel to a given straight line.

EXERCISES.

1. Any two angles of an acute-angled triangle are greater than the third.

2. If an angle of a triangle be greater than the sum of the other two, it is obtuse.

3. Each of the angles at the base of an isosceles triangle is half of the exterior angle at the vertex.

4. CD is drawn perpendicular to AB in the triangle ABC; prove that the difference of the angles DCA, DCB is equal to the difference of the angles CBA, BAC.

5. ABC is a triangle; from A are drawn AD, AE, making the angles BAD, CAE respectively equal to the angles ACB, ACE, and meeting BC at D, E: prove that ADE is an isosceles triangle.

6. The angle between the bisector of one of the base angles of a triangle and the bisector of the exterior angle adjacent to the other base angle is equal to half the vertical angle of the triangle.

7. If one angle of a triangle be three times another angle, shew that the triangle can be divided into two isosceles triangles.

8. The angle BAC of the triangle ABC is bisected by AD meeting BC in D; if DC is equal to DA, prove that the angle BDA is double of the angle BAD.

9. ABC is an isosceles triangle having the angle at B four times each of the other angles; AB is produced to D such that BD is equal to twice AB, and CD is joined. Shew that the triangles ACD, ABC are equiangular to one another.

10. BC is the greatest side of the triangle ABC, and CD is cut off from it equal to CA, and AD is joined. Prove that the difference of the angles into which the angle CAB is divided by AD is equal to the angle ABC.

11. ABC is a triangle, and from A, B, C are drawn AD, BE, CF, making the angles BAD, CBE, ACF equal to one another. If AD, BE, CF do not meet in a point, prove that they form a triangle, the angles of which are respectively equal to those of the triangle ABC.

12. ABC is a right-angled triangle and CD is drawn perpendicular to AB from the right angle C. From BA cut off BG equal to BC, and prove that GC bisects the angle DCA.

13. If one of the equal sides of an isosceles triangle be bisected at D and also be doubled by being produced through the extremity of the base to E, then the distance of the other extremity of the base from E is double its distance from D.

EXERCISES (*continued*).

14. If two angles of a triangle are equal respectively to two angles of another triangle, the third angles of the triangles are equal.

15. What part of a right angle is an angle of an equilateral triangle?

16. Trisect (i) a right angle; (ii) half a right angle; (iii) one-fourth of a right angle; (iv) two right angles.

17. If one of the acute angles of a right-angled triangle be one-third of a right angle, prove that it is one-half of the other acute angle.

18. The angles of a triangle are to one another as 1 : 2 : 3. Prove that the triangle is right-angled.

19. The bisectors of two angles of a triangle contain an obtuse angle.

20. In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of it are equiangular to the whole triangle and to one another.

21. If A, B, C be the angles of a triangle, $\frac{A+B}{2}$, $\frac{B+C}{2}$, $\frac{C+A}{2}$ will be the angles of a triangle formed by any side and the bisectors of the exterior angles between that side and the other sides produced.

22. The interior angles of any quadrilateral are together equal to four right angles.

23. If the opposite angles of a four-sided figure be equal, the figure is a parallelogram.

24. Describe a rhombus having one angle double of another.

25. From the extremities of the base of an isosceles triangle are drawn perpendiculars to the sides; prove that the angles made by them with the base are each equal to half the vertical angle.

26. On the sides of a triangle ABC are described equilateral triangles AFB, BDC, CEA, all external; prove that AD, BE, CF are equal.

27. ACB is a right-angled triangle; on the hypotenuse AB are taken AD equal to AC, and BE equal to BC; shew that ECD is half a right angle.

28. ABC is an equilateral triangle, and BC is produced both ways to D and E so that BD and CE are each equal to BC. Join AD, AE, and draw CFG at right angles to DE meeting AE at F and DA produced at G. Prove that AFG is an equilateral triangle.

29. ABC is an equilateral triangle; through A, B and C draw lines at right angles to AC, BA, CB respectively; prove that these lines form another equilateral triangle.

30. S, E and W start from the same spot, S going south at half the pace of each of the others, whose directions on opposite sides make with the course of S angles each equal to two-thirds of a right angle. Shew that the line joining W, S and E at any moment is a straight line running direct east and west.

31. Two straight lines which intersect at P are respectively perpendicular to two straight lines which intersect at Q; shew that the inclination of the first two lines is equal to that of the others.

32. D and E are the middle points of the sides AB, AC of the triangle ABC; CD and BE are produced to F and G so that DF is equal to DC, and EG to BE. FAG shall be a straight line.

EXERCISES (*continued*).

33. The angle contained by the bisector of the vertical angle of any triangle and the perpendicular from the vertex on the base, is equal to half the difference of the base angles.

34. AEB, CED are two straight lines intersecting at E; straight lines AC, DB are drawn forming two triangles ACE, BED; CF, BF bisect the angles ACE, DBE. Shew that the angle CFB is half the sum of the angles EAC, EDB.

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35. The median from the vertex of a triangle is greater than, equal to, or less than half the base, according as the vertical angle is acute, right, or obtuse.

[Vide Ex. 12, Prop. XXVII. AE and BC being parallel, the angle BAE is obtuse, right, or acute, according as the angle ABC is acute, right, or obtuse; and the result readily follows from Props. XXIV. and IV.]

36. If DE, DF be the equal sides of an isosceles triangle, and ED be produced to G so that DG is equal to DE; EFG is a right angle.

37. Draw a straight line at right angles to a given straight line from its extremity without producing the given line.

38. In the triangle PQR, PQ is equal to PR; QM is drawn at right angles to QR to meet RP produced in M; shew that PM is equal to PR.

39. If a triangle has one angle equal to the sum of the other two, it is right-angled, and can be divided into two isosceles triangles.

40. From D the middle point of AB, DC is drawn in any direction and equal to AD; prove that ACB is a right angle.

41. From B, the right angle of the triangle ABC, BD is drawn meeting the hypotenuse in D; shew that if DA is equal to DB, then DC is also equal to DB.

42. ABC is a triangle; AD is perpendicular to BC, and AB, AC are bisected in E, F; shew that the angle EDF is equal to the angle BAC.

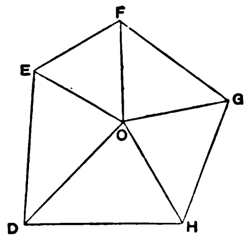
43. ABC is a triangle; BD, CE are perpendiculars on the sides AC, AB, produced if necessary; from G, the middle point of BC, GF is drawn perpendicular to DE. Prove that EF is equal to FD.

44. AD, BC are two parallel straight lines, cut obliquely by AB and perpendicularly by AC; BED is drawn cutting AC in E and AD in D such that ED is equal to twice BA. Prove that the angle DBC is one-third the angle ABC.

45. On the hypotenuse BC of a right-angled triangle ABC is described externally an isosceles right-angled triangle BCD; prove that AD bisects the angle BAC.

46. Prove that the point of intersection of the diagonals of a square described on the hypotenuse of a right-angled triangle is equidistant from the two sides containing the right angle.

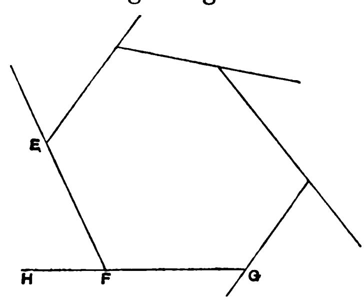
Corollary 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.



For any rectilineal figure *DEFGH* can be divided into as many $\triangle$s as the figure has sides, by drawing st. lines from a point *O* within the figure to each of its angles.

Then, $\because$ the intr. $\angle$s of a $\triangle = 2$ rt. $\angle$s,
all the $\angle$s of the $\triangle$s = twice as many rt. $\angle$s as there are $\triangle$s
= twice as many rt. $\angle$s as the figure has sides.
Again, all the $\angle$s of the $\triangle$s = the intr. $\angle$s of the figure, together with the $\angle$s at *O*, which is the common vertex of the $\triangle$s
= the intr. $\angle$s of the figure, together with 4 rt. $\angle$s. $\left\{\begin{array}{l}\textit{Cor.}\ 2.\\ \text{I. xv.}\end{array}\right.$
$\therefore$ the intr. $\angle$s of the figure, together with 4 rt. $\angle$s.
= twice as many rt. $\angle$s as the figure has sides. *Ax.* 1.

Corollary 2. All the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles.



$\because$ every intr. $\angle$ *EFG*, with its adjacent extr. $\angle$ *EFH* = 2 rt. $\angle$s, I. XIII.
$\therefore$ all the intr. $\angle$s, with all the extr. $\angle$s = twice as many rt. $\angle$s as the figure has sides.
But the intr. $\angle$s, with 4 rt. $\angle$s = twice as many rt. $\angle$s as the figure has sides; *Cor.* 1.
$\therefore$ the intr. $\angle$s, with all the extr. $\angle$s = the intr. $\angle$s, with 4 rt. $\angle$s; *Ax.* 1.
$\therefore$ all the extr. $\angle$s = 4 rt. $\angle$s. *Ax.* 3.

This corollary is true only when the figure is convex.

REFERENCES.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 3. If equals be taken from equals the remainders are equal.

I. XIII. The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

I. XV. Cor. 2. All the angles made by any number of straight lines meeting at one point, are together equal to four right angles.

EXERCISES.

47. A convex polygon cannot have more than three of its interior angles acute.

48. To how many right angles are the interior angles of an irregular octagon equal?

49. By how much does the angle of a regular nonagon exceed a right angle?

50. The angle of a regular decagon is four thirds the angle of a regular pentagon.

51. For the floor of a church I am recommended three tiles:—a square, a regular hexagon, and a regular dodecagon. Will the angles fit? Will equilateral triangles and regular hexagons do for the purpose?

52. What is the polygon each of whose angles is:

(i) ten-sevenths of a right angle; (ii) eighteen-elevenths of a right angle?

53. Two regular polygons of the same number of sides are equal in all respects, if a side of one is equal to a side of the other.

54. The alternate sides of a polygon are produced to meet; shew that the angles at their points of intersection, together with four right angles, are equal to all the interior angles of the polygon.

55. If the sides of a regular hexagon be produced to meet, the angles at the points of meeting are together equal to four right angles.

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56. Each of the exterior angles of a regular polygon is half a right angle; shew that the polygon is an octagon.

57. Each of the exterior angles of a regular polygon is equal to half one of the angles of an equilateral triangle; how many sides has the polygon?

58. The sum of the exterior angles of a polygon is equal to half the sum of the interior angles. What is the polygon?

59. A convex polygon cannot have more than three obtuse exterior angles.

60. The bisectors of the exterior angles of a quadrilateral form another quadrilateral, the sum of each pair of opposite angles of which is equal to two right angles.

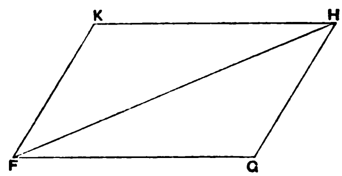
61. If the sides of a heptagon be produced both ways, the alternate sides meeting and forming a starlike figure, shew that the angles at the points of the star are together equal to six right angles.

PROPOSITION XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let KH, FG be equal and parallel st. lines, and let them be joined towards the same parts by the st. lines, KF, HG.

KF, HG shall be equal and parallel.



Join HF. *Post.* 2.

$\because$ KH is $\parallel$ to FG, and HF meets them, *Hyp.*

the $\angle$ KHF = the alternate $\angle$ HFG. I. XXIX.

Then $\because$ $KH = FG$, *Hyp.*

and HF is common to the two $\triangle$s KHF, GFH;

the 2 sides KH, HF = the 2 sides GF, FH, each to each,

and the $\angle KHF$ was shewn = the $\angle HFG$;

$\therefore$ the base KF = the base GH, I. IV.

and the $\triangle$ KHF = the $\triangle$ GFH,

and the other $\angle$s = the other $\angle$s, each to each, to which the equal sides are opposite;

$\therefore$ the $\angle KFH$ = the $\angle FHG$.

And $\because$ the st. line HF meets the two st. lines FK, HG, and makes the alternate $\angle$s KFH, FHG equal to one another,

KF is $\parallel$ to HG; I. XXVII.

and KF has been shewn = HG.

Wherefore *the straight lines &c.* Q. E. D.

REFERENCES.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. XXVII. If a straight line falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

I. XXIX. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

EXERCISES.

1. The straight line joining the bisections of two opposite sides of a rhombus is parallel to the other sides.

2. The straight lines joining the extremities of two unequal parallel straight lines towards the same parts will meet if produced on the side of the shorter parallel.

3. The straight lines which join the extremities of two equal and parallel straight lines, but not towards the same parts, bisect each other.

4. The extremities of two equal and parallel straight lines AB, CD are joined; prove that, if BC be greater than AD, the angle BDC is obtuse, and that the bisectors of the angles ABD, BDC are at right angles.

5. Triangles and parallelograms between the same parallels have the same altitude; and triangles and parallelograms of the same altitude may be placed between the same parallels.

6. $ABEF$, $EBCD$ are two parallelograms on opposite sides of a common base EB; join AC and FD, and prove that $ACDF$ is a parallelogram.

7. AB, CD, EF are three equal and parallel straight lines. Prove that the triangles ACE, BDF, formed by joining their extremities towards the same parts, are equal in all respects.

8. If two straight lines MN, NO be respectively equal and parallel to the two straight lines PQ, QR, prove that MO is equal and parallel to PR.

9. A, B, C are the centres of three equal circles intersecting at one point O. The circles, whose centres are A, C, also intersect at P, and the circles, whose centres are B, C, at Q. Prove that $APQB$ is a parallelogram.

10. AB, CD are two equal and parallel straight lines. From A and B are drawn two straight lines parallel to each other, and on these are let fall perpendiculars from C and D; prove that the perpendiculars are equal.

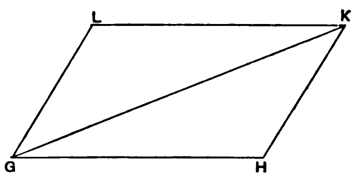
11. In a straight line AD are taken four points A, B, C, D, such that AB is equal to CD. From B and D are drawn two equal straight lines BE, DF at right angles to AD. Prove that $ACFE$ is a parallelogram.

PROPOSITION XXXIV. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram, that is, divides it into two equal parts.

Let $GHKL$ be a ||gram, of which GK is a diameter.

The opposite sides and $\angle$s of the figure shall = one another; and the diameter GK shall bisect it.



$\because LK$ is || to GH, and GK meets them,
the $\angle LKG$ = the alternate $\angle KGH$. I. XXIX.
and $\because LG$ is || to KH, and GK meets them,
the $\angle LGK$ = the alternate $\angle GKH$. I. XXIX.
$\therefore$ in the two $\triangle$s LKG, HGK,
the 2 $\angle$s LKG, KGL in the one = the 2 $\angle$s HGK, GKH in the other, each to each,
and the side KG, adjacent to their equal $\angle$s, is common to the two $\triangle$s;
$\therefore$ their other sides are equal, each to each, and the third $\angle$ of the one to the third $\angle$ of the other, I. XXVI.
namely, the side LK to the side GH, and LG to KH, and the $\angle KLG$ to the $\angle GHK$.
And $\because$ the $\angle LKG$ = the $\angle HGK$,
and the $\angle GKH$ = the $\angle KGL$;
$\therefore$ the whole $\angle LKH$ = the whole $\angle HGL$; *Ax.* 2.
and the $\angle KLG$ has been shewn = the $\angle GHK$.
$\therefore$ the opposite sides and $\angle$s of a || gram = one another.
Also the diameter bisects the || gram.
For, $\because LK = HG$, and GK is common to the two $\triangle$s LKG, HGK,
the 2 sides LK, KG = the 2 sides HG, GK, each to each,
and the $\angle LKG$ has been shewn = the $\angle HGK$;
$\therefore$ the $\triangle LKG$ = the $\triangle HGK$, I. IV.
and the diameter GK divides the || gram $GHKL$ into two equal parts.

Wherefore *the opposite sides &c.* Q. E. D.

REFERENCES.

Ax. 2. If equals be added to equals the wholes are equal.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. XXVI. If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

I. XXIX. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

EXERCISES.

1. The diagonals of a rectangle are equal. Also prove the converse.

2. If a parallelogram has two sides about one of its angles equal, it is a rhombus.

3. If P, Q, R, S are the middle points of the sides of a rectangle, $PQRS$ is a rhombus.

4. If two sides of a quadrilateral be parallel and unequal, and the other two sides equal but not parallel, its opposite angles are supplementary.

5. The perimeter of the parallelogram formed by drawing parallels to two sides of an equilateral triangle from any point in the base is double a side of the triangle.

6. The perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides are together equal to the perpendicular from one extremity of the base to the opposite side.

7. $ABCD$ is a parallelogram; from A is drawn a line AE cutting BC. Prove that the distance of C from this line is equal to the difference of the distances of B and D.

8. The bisectors of the acute angles, CAB, ABC, of a right-angled triangle ABC, meet at O; OP, OQ are perpendiculars on AC, CB. Prove that $OPCQ$ is a square.

9. The straight line drawn through the middle point of one side of a triangle parallel to a second side bisects the third side.

10. The angle B of the triangle ABC is bisected by BD to which a perpendicular AD is drawn from A; DE is drawn parallel to BC meeting AC in E. Prove that AE is equal to EC, and that DE is half the difference between AB and BC.

11. If in the diagonal of a parallelogram any two points equidistant from its extremities be taken and joined with the opposite angles, a parallelogram will be formed.

12. $ABCD$ is a quadrilateral having the sides AD, BC parallel; AF is drawn parallel to DC meeting BC at F, and DE parallel to AB meeting BC at E. Prove that the triangles ABF, DEC are equal in all respects.

13. Place a straight line of given length between two straight lines which meet, so that it shall be equally inclined to them.

14. Draw a line terminated by two given straight lines, such that it shall be equal to one and parallel to another given straight line.

To divide a straight line into any number of equal parts.

1. $ABCD$ is a parallelogram; E, F are the middle points of AD, BC respectively; prove that BE and CF trisect the diagonal AC.

2. AE and CG are equal parallel straight lines; produce them, and on the parts produced take EF, FB, each equal to AE, and GH, HD, each equal to CG; join BC, and draw the straight lines AG, EH, FD, cutting CB at K, L and M respectively. Shew that CB is divided into four equal parts at K, L and M. Also shew how the method may be employed to divide a straight line into any given number of equal parts.

3. Let AB, AC be two straight lines meeting at A; on AB produced take BD, DE, each equal to AB; join EC, and through B and D draw BF, DG parallel to EC and meeting AC in F and G. Shew that AC is trisected in F and G. Also prove that BF is one-third, and DG two-thirds of EC.

4. Employ the method of the last deduction to divide a given straight line into five equal parts.

5. C is the middle point of AB; AD, CF, BE are drawn perpendicular to AB, on the same side of it; on AD are taken ten equal distances Ad_1, d_1d_2, d_2d_3,......d_9D; through the points of section d_1, d_2,...... are drawn parallels to AB, cutting CF and BE; CD is joined, and the parallel through d_6 meets CD in G and BE in H. If AB is 20 inches, what is the length of GH? If AB is 2 feet, indicate by thick lines lengths of ·9 and 1·3 feet.

This shews the method of making diagonal scales.

MISCELLANEOUS EXERCISES.

1. No straight line can be placed in a parallelogram greater than the greater diameter.

2. Through G, a point outside the triangle ABC, GE is drawn parallel to AC and GD parallel to BC; through A and B are drawn AE, BD parallel to CG meeting GE and GD respectively in E and D. Prove that the triangle DEG is identically equal to the triangle ABC.

3. If the straight line joining two opposite angles of a parallelogram bisect the angles, the parallelogram is equilateral.

4. C is the middle point of the line AB; prove that the sum of the perpendiculars from A and B on any line which does not intersect the finite line AB is double of the perpendicular from C on the same line.

5. From the angles of any parallelogram draw perpendiculars to any straight line outside the parallelogram, and shew that the sum of those from one pair of opposite angles is equal to the sum of those from the other pair of opposite angles.

6. There are seats at regular intervals on both sides of a straight avenue. Sitting on one of these I heard the cries of a child in a perambulator some distance off in a direct line with the last seat on the other side. I walked in that direction on my own side and on reaching the next seat observed the child still crying in a line with the nurse-maid who was occupying the last seat but one on the other side. Shew that the perambulator was in the middle of the avenue.

7. Find points D and E in the equal sides AB, AC of an isosceles triangle, such that BD, DE, EC are equal to one another.

8. If a straight line joining the extremities of two straight lines which are equal, but not parallel, make the angles on one side of it equal to one another, it will be parallel to the straight line joining the other extremities.

9. From the extremities A and B of the straight line AB as centres and with radius AB circles are described intersecting in C and D; CA and CB are produced to meet the circles in E and F; prove that the straight line EF passes through D.

10. In the figure of Prop. I. the given line is produced to meet either of the circles in P; shew that P and the points of intersection of circles are the angular points of an equilateral triangle.

11. ABC is a triangle having B a right angle, and the angle A is double of the angle C; BD is drawn perpendicular to AC, and BE is drawn to E the middle point of AC; shew that the angle DBE is equal to the angle ACB.

12. Prove that, on a giant stride, if the hands are kept at the same distance from the ground, they describe a circle.

13. ABC is an isosceles triangle having AB equal to AC; P is any point in AC, and PQR is drawn meeting BC in Q and AB produced in R, and such that PQ is equal to QR. Prove that the sum of AP and AR is equal to the sum of AB and AC.

14. ABC is an isosceles triangle whose equal sides are AB, AC; PQ is a straight line cutting the equal sides in P, Q, so that the sum of AP and AQ is equal to the sum of the equal sides: prove that PQ is greater than BC.

15. If the opposite sides of a hexagon be equal and parallel, its diagonals are concurrent.

16. Two straight lines which intersect at right angles and are terminated by the opposite sides of a square are equal to one another.

17. If the areas of two equilateral triangles be equal, their sides are equal.

18. The perpendiculars from the vertices of a triangle on the opposite sides are concurrent.

[Draw parallels to the opposite sides through the vertices, which prove to be the middle points of the sides of the triangle thus formed, and use Ex. 13, Prop. XXIX.]

19. AD, AE are squares on the sides AB, AC of a triangle ABC, and DF, EG are perpendiculars on the base BC produced. Prove that DF, EG are together equal to BC.

SECTION III.

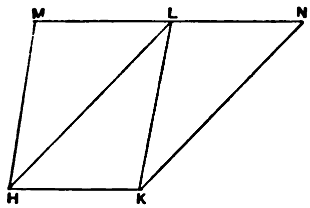
PROPOSITION XXXV. THEOREM.

Parallelograms on the same base, and between the same parallels, are equal to one another.

Let the ||grams $MHKL$, $PHKN$ be on the same base HK, and between the same parallels MN, HK.

The ||gram $MHKL$ shall = the ||gram $PHKN$.

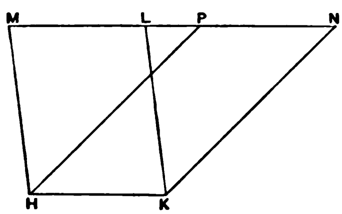
If the sides ML, LN of the ||grams $MHKL$, $LHKN$, opposite to the base HK, be terminated at the same point L;

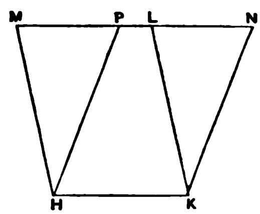


each of the ||grams is double of the $\triangle LHK$; I. XXXIV.

and $\therefore$ the ||gram $MHKL$ = the ||gram $LHKN$. *Ax.* 6.

But if the sides ML, PN of the ||grams $MHKL$, $PHKN$, opposite to the base HK, be not terminated in the same point;





then, $\because$ $MHKL$ is a ||gram,

$ML = HK$. I. XXXIV.

Similarly, $PN = HK$;

$\therefore$ $PN = ML$; *Ax.* 1.

$\therefore$ the whole, or the remainder, LN = the whole, or the remainder, MP; *Ax.* 2, 3.

and $LK = MH$; I. XXXIV.

$\therefore$ the 2 sides NL, LK = the 2 sides PM, MH, each to each;

and the extr. $\angle NLK$ = the intr. and opposite $\angle PMH$; I. XXIX.

$\therefore$ the $\triangle NLK$ = the $\triangle PMH$. I. IV.

From the trapezium $MHKN$ take the $\triangle NLK$;

and again, from the same trapezium $MHKN$ take the $\triangle PMH$;

and the remainders are equal; *Ax.* 3.

that is, the ||gram $MHKL$ = the ||gram $PHKN$.

Wherefore *parallelograms on the same base &c.* Q. E. D.

The word "equal" means equal in area, and is no longer restricted to mean *coincidence in form.* The parallelograms MHKL, PHKN are equal in area but, *if they were applied* one to the other, their boundaries would not necessarily *coincide.*

REFERENCES.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 2. If equals be added to equals the wholes are equal.

Ax. 3. If equals be taken from equals the remainders are equal.

Ax. 6. Things which are double of the same thing are equal to one another.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. XXIX. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

I. XXXIV. The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram.

EXERCISES.

1. Prove the converse theorem :—

Equal parallelograms on the same base, and on the same side of it, are between the same parallels.

2. Of two parallelograms on the same base, and on the same side of it, that is the greater of which the altitude is the greater.

3. Describe a parallelogram equal to a given parallelogram, and having an angle equal to a given angle.

4. On either diagonal of a square construct a rhombus equal in area to the square.

5. Prove that a square has a greater area than a rhombus of the same perimeter.

6. AB and ECF are two parallel straight lines. Through B are drawn BC, BD parallel respectively to AE, AF, meeting ECF in C and D. Prove that the triangle BFD is equal to the triangle ABC.

7. If through each vertex of a triangle a straight line be drawn parallel to the opposite side, the triangle thus formed will have three equal parallelograms inscribed in it, and its area will be four times that of the original triangle.

8. The area of a quadrilateral is equal to the area of a triangle having two sides equal to the diagonals, and the contained angle equal to that between the diagonals.

9. Prove that the area of a parallelogram is the product of the base into the altitude.

10. The sides AB, AD of a rectangle $ABCD$ are respectively 10 ft. and 9 ft. Join AC, produce DC to E, and through B draw BE parallel to AC. Find the area of the figure $ABEC$.

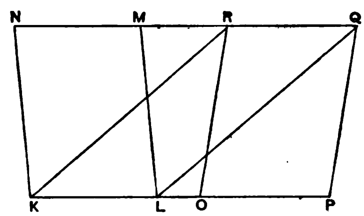
11. If the bases of two equal parallelograms are 35 and 42 inches, and the altitude of the former is 12 inches, what is the altitude of the other?

PROPOSITION XXXVI. THEOREM.

Parallelograms on equal bases, and between the same parallels, are equal to one another.

Let $NKLM$, $ROPQ$ be ||grams on equal bases KL, OP, and between the same parallels NQ, KP.

The ||gram $NKLM$ shall = the ||gram $ROPQ$.



Join KR, LQ.	*Post.* 1.
Then, $\because KL = OP$,	*Hyp.*
and $RQ = OP$;	I. XXXIV.
$\therefore KL = RQ$;	*Ax.* 1.
and they are parallels,	*Hyp.*

and joined towards the same parts by the st. lines KR, LQ.

$\therefore KR$, LQ are both equal and parallel. I. XXXIII.

$\therefore RKLQ$ is a ||gram.

And $\because$ the ||grams $NKLM$, $RKLQ$ are on the same base KL, and between the same parallels KL, NQ,

$\therefore$ the ||gram $NKLM$ = the ||gram $RKLQ$. I. XXXV.

For the same reason, the ||gram $ROPQ$ = the ||gram $RKLQ$.

$\therefore$ the ||gram $NKLM$ = the ||gram $ROPQ$. *Ax.* 1.

Wherefore *parallelograms &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 1. Things which are equal to the same thing are equal to one another.

I. XXXIII. The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

I. XXXIV. The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram.

I. XXXV. Parallelograms on the same base, and between the same parallels, are equal to one another.

EXERCISES.

1. Equal parallelograms between the same parallels have equal bases.

2. Of two parallelograms between the same parallels, the greater has the greater base; and conversely, that which has the greater base is greater than the other.

3. If two parallelograms are on opposite sides of a common base and have the same altitude, they are equal.

4. E and F are the middle points of the sides BC, CD of the parallelogram $ABCD$; EG drawn parallel to AB, and FG drawn parallel to AD meet at G: prove that the parallelogram $ECFG$ is one-fourth the parallelogram $ABCD$.

5. Divide a parallelogram into three equal parallelograms.

6. All straight lines passing through the intersection of the diagonals of a parallelogram and terminated by the sides are bisected at the point of intersection and bisect the parallelogram.

7. In the side of a parallelogram take a point (not the middle point) and through it draw a straight line which shall bisect the parallelogram.

8. Through any point within a parallelogram draw a straight line to bisect the parallelogram.

9. If $GHKL$ be a quadrilateral having the side LK parallel to GH, prove that it is equal to a parallelogram formed by drawing through the middle point of HK a straight line parallel to GL to meet the sides GH, LK.

10. E is the middle point of BC, a side of the quadrilateral $ABCD$, in which AB is parallel to CD: prove that the quadrilateral is double of the triangle AED.

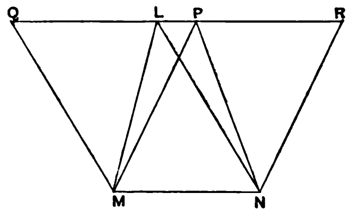
11. On AB, BC, sides of the triangle ABC, are described the parallelograms $ABDE$, $BCFG$, and the sides ED, FG, produced if necessary, meet at H. On AC is described a parallelogram $ACKL$, whose opposite sides CK, AL are equal and parallel to BH. Prove that the parallelogram AK is equal to the sum of the parallelograms CG, BE.

PROPOSITION XXXVII. THEOREM.

Triangles on the same base, and between the same parallels, are equal.

Let the $\triangle$s LMN, PMN be on the same base MN, and between the same parallels LP, MN.

The $\triangle LMN$ shall $=$ the $\triangle PMN$.



Produce LP both ways to the points Q, R; *Post.* 2.

through M draw $MQ \parallel$ to NL, and through N draw NR $\parallel$ to MP. I. XXXI.

Then each of the figures $QMNL$, $PMNR$ is a $\parallel$gram.

And $QMNL = PMNR$,

$\because$ they are on the same base MN, and between the same parallels MN, QR. I. XXXV.

And the $\triangle LMN$ is half of the $\parallel$gram $QMNL$,

$\because$ the diameter LM bisects the $\parallel$gram. I. XXXIV.

Also the $\triangle PMN$ is half of the $\parallel$gram $PMNR$,

$\because$ the diameter PN bisects the $\parallel$gram. I. XXXIV.

But the halves of equal things are equal. *Ax.* 7.

$\therefore$ the $\triangle LMN =$ the $\triangle PMN$.

Wherefore *triangles &c.* Q. E. D.

REFERENCES.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

Ax. 7. Things which are halves of the same thing are equal to one another.

I. xxxi. To draw a straight line through a given point parallel to a given straight line.

I. xxxiv. The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram.

I. xxxv. Parallelograms on the same base, and between the same parallels, are equal to one another.

EXERCISES.

1. Describe a triangle equal to a given triangle and having an angle equal to a given angle.

2. The straight line PQ is drawn parallel to NO, a side of the triangle MNO, and meets MN, MO at P and Q. Shew that the triangle NPO is equal to the triangle NQO, and that the triangle MNQ is equal to the triangle MOP.

3. The side CB of the triangle ABC is produced to any point D, and through B is drawn BE parallel to AD meeting AC in E. Prove that the triangle DEC is equal to the triangle ABC.

4. The diagonals AC, BD of the quadrilateral $ABCD$ meet at E. If AB is parallel to CD, shew that the triangle AED is equal to the triangle BEC.

5. P is any point in AB, a side of the triangle ABC. In BC produced find a point Q such that the triangle BPQ may be equal to the triangle ABC.

6. Upon a given base describe a triangle equal to a given triangle.

7. Describe a triangle equal to a given triangle and having a given altitude.

8. ABC is any triangle; on BC describe an isosceles triangle equal to ABC.

9. Describe a triangle equal to any quadrilateral.

10. Describe a triangle equal to any pentagon.

11. Enunciate the converse of the theorem:—If a quadrilateral be a parallelogram, its diameter bisects it. Prove that the converse theorem is not true.

12. Describe a rhombus equal to a given parallelogram.

13. Of all triangles having two sides equal to two given straight lines, that is the greatest in which these sides are at right angles to each other.

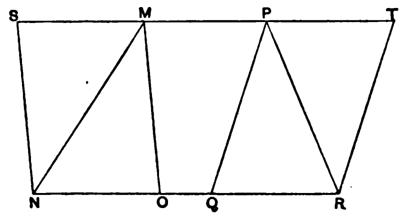
14. A rhombus is greater than any other parallelogram having the same unequal straight lines for diagonals.

PROPOSITION XXXVIII. THEOREM.

Triangles on equal bases, and between the same parallels, are equal to one another.

Let the $\triangle$s MNO, PQR be on equal bases NO, QR, and between the same parallels NR, MP.

The $\triangle MNO$ shall = the $\triangle PQR$.



Produce MP both ways to the points S, T; *Post.* 2.
through N draw NS || to OM, and through R draw RT || to QP. I. XXXI.

Then each of the figures $SNOM$, $PQRT$ is a ||gram.
And $SNOM = PQRT$,
$\because$ they are on equal bases NO, QR, and between the same parallels NR, ST. I. XXXVI.
And the $\triangle MNO$ is half of the ||gram $SNOM$,
$\because$ the diameter MN bisects the ||gram. I. XXXIV.
Also the $\triangle PQR$ is half of the ||gram $PQRT$,
$\because$ the diameter PR bisects the ||gram. I. XXXIV.
But the halves of equal things are equal. *Ax.* 7.
$\therefore$ the $\triangle MNO$ = the $\triangle PQR$.

Wherefore *triangles &c.* Q. E. D.

REFERENCES.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

Ax. 7. Things which are halves of the same thing are equal to one another.

I. XXXI. To draw a straight line through a given point parallel to a given straight line.

I. XXXIV. The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram.

I. XXXVI. Parallelograms on equal bases, and between the same parallels, are equal to one another.

EXERCISES.

1. D is any point in BC, a side of the triangle ABC, and AB, AC are bisected in E and F; prove that the figure $AEDF$ is half the triangle ABC.

2. If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles supplementary, they are equal in area.

3. The diagonals divide a parallelogram into four equal triangles.

4. If two sides of a quadrilateral be parallel to one another, the straight line joining their bisections bisects the quadrilateral.

5. Bisect a triangle by a line drawn from a given point in a side.

6. If in the triangle ABC, BC be bisected in D, AD joined and bisected in E, BE joined and bisected in F, CF joined and bisected in G, the triangle EFG is one-eighth of the triangle ABC.

7. $ABCD$ is a parallelogram; P and Q, the middle points of the sides BC, CD: shew that the triangle APQ is equal to the sum of the triangles BPQ, CPQ, DPQ, and is $\frac{3}{8}$ths of the whole parallelogram.

8. The straight lines AD, BE, bisecting the sides BC, CA of the triangle ABC, intersect at G; shew that the triangles EGA, BGD, CGD, CGE are each one-sixth of the triangle ABC, and that the triangle AGB is one-third the triangle ABC.

9. OPQ is any triangle; OR bisects PQ in R, PST bisects OR in S and meets OQ in T: prove that the triangle OPQ is equal to three times the triangle TPR, or four times the triangle SQR, or twelve times the triangle OST.

10. $ABCD$ is a quadrilateral, having AB parallel to DC and less than it; find a point E in DC, such that the triangle DBE may be half the figure $ABCD$.

11. From the sides AB, BC, CA of an equilateral triangle, AM, BN, CO are cut off each equal to one-third one of the sides of the triangle; shew that the triangle MNO is one-third the triangle ABC.

12. Describe a triangle two-thirds as large again as a given triangle.

13. Given a triangle, describe another such that four times the latter may be equal to fifteen times the given triangle.

14. ABC is a triangle; D and E are taken on AB, AC, such that AD is one-third of AB, and AE one-third of AC; CD, BE intersect at F. Prove that the triangles DEF, AEF, AED, DFB, EDC, BFC are respectively $\frac{1}{18}$th, $\frac{1}{12}$th, $\frac{1}{9}$th, $\frac{1}{6}$th, $\frac{2}{9}$ths and half of the triangle ABC.

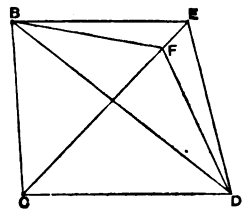
15. Bisect a given quadrilateral by a line drawn through one of its angular points.

PROPOSITION XXXIX. THEOREM.

Equal triangles on the same base, and on the same side of it, are between the same parallels.

Let the equal $\triangle$s BCD, ECD be on the same base CD, and on the same side of it.

They shall be between the same parallels.



Join BE. *Post.* 1.

BE shall be $\parallel$ to CD.

For if not, through B draw $BF \parallel$ to CD, meeting CE at F, and join FD. I. XXXI. *Post.* 1.

Then the $\triangle BCD =$ the $\triangle FCD$,

$\because$ they are on the same base CD, and between the same parallels CD, BF. I. XXXVII.

But the $\triangle BCD =$ the $\triangle ECD$. *Hyp.*

$\therefore$ the $\triangle ECD =$ the $\triangle FCD$, *Ax.* 1.

the greater to the less, which is impossible.

$\therefore$ BF is not $\parallel$ to CD.

In the same manner it can be shewn,

that no other line through B but BE is $\parallel$ to CD.

$\therefore$ BE is $\parallel$ to CD.

Wherefore *equal triangles &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 1. Things which are equal to the same thing are equal to one another.

I. XXXI. To draw a straight line through a given point parallel to a given straight line.

I. XXXVII. Triangles on the same base, and between the same parallels, are equal.

EXERCISES.

1. Any quadrilateral which is bisected by both of its diagonals is a parallelogram.

2. AC and BD are the diagonals of a quadrilateral $ABCD$; if the triangle ABD is equal to the triangle ABC, prove that the triangle ADC is equal to the triangle BDC.

3. E and F are two points within the triangle ABC; if the sum of the triangles AEB, BEC is equal to the sum of the triangles AFB, BFC, prove that EF is parallel to AC.

4. If of the four triangles into which the diagonals divide a trapezium any two opposite ones are equal, the trapezium has a pair of opposite sides parallel.

[The name trapezium is by some writers used only for a quadrilateral which has one pair of opposite sides parallel.]

5. If two sides of a triangle be bisected, the straight line which joins the points of bisection is parallel to the third side, and equal to the half of it.

6. If a straight line be drawn from the vertex of a triangle ABC to any point in the base BC, prove that the middle points of AB, AD, AC are in a straight line.

7. ABC, DBC are two triangles on the same base BC, and on the same side of it, but not between the same parallels. The sides AB, AC are bisected at E and F, and the sides DB, DC, at G and H. Prove that $EFHG$ is a parallelogram.

8. Construct a triangle, having given the middle points of the sides.

9. Divide a given triangle into four identically equal triangles.

10. The perimeter of an isosceles triangle is less than that of any other triangle of equal area, standing on the same base.

11. If D, E, F be the middle points of the sides BC, CA, AB of the triangle ABC; BE, CF, AD are concurrent, and divide each other in the ratio of $2 : 1$.

[Let BE, CF intersect at G; join AG and prove that when produced it will bisect BC. Through B draw BH parallel to FC to meet AG produced at H; shew that $BGCH$ is a parallelogram, using Ex. 9, Prop. XXXIV. and Ex. 5 above.]

12. If a diagonal bisects a quadrilateral, it also bisects the other diagonal. Also prove the converse.

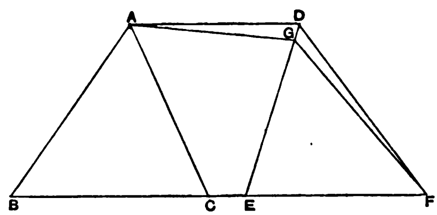
13. If the diagonals of a quadrilateral $ABCD$ intersect at E, and the triangle BCD be double of the triangle BAD; EC is double of EA.

PROPOSITION XL. THEOREM.

Equal triangles, on equal bases in the same straight line, and on the same side of it, are between the same parallels.

Let the equal $\triangle$s ABC, DEF be on equal bases BC, EF, in the same st. line BF, and on the same side of it.

They shall be between the same parallels.



Join AD. *Post.* 1.

AD shall be $\parallel$ to BF.

For if not, through A draw $AG \parallel$ to BF, meeting ED at G, I. XXXI.

and join GF. *Post.* 1.

Then the $\triangle ABC =$ the $\triangle GEF$,

$\because$ they are on equal bases BC, EF, and between the same parallels BF, AG. I. XXXVIII.

But the $\triangle ABC =$ the $\triangle DEF$, *Hyp.*

$\therefore$ the $\triangle DEF =$ the $\triangle GEF$, *Ax.* 1.

the greater to the less; which is impossible.

$\therefore AG$ is not $\parallel$ to BF.

In the same manner it can be shewn, that no other line through A but AD is $\parallel$ to BF.

$\therefore AD$ is $\parallel$ to BF.

Wherefore *equal triangles &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 1. Things which are equal to the same thing are equal to one another.

I. XXXI. To draw a straight line through a given point parallel to a given straight line.

I. XXXVIII. Triangles on equal bases, and between the same parallels, are equal to one another.

EXERCISES.

1. Equal triangles on equal bases have the same altitude.

2. Three equal triangles are on equal bases, in the same straight line, and on the same side of it; prove that their vertices, if no two of them be coincident, are in one straight line.

3. If two equal triangles ABC, DEF be on equal bases AB, EF, in the same straight line AF, but on opposite sides of it; the line CD joining their vertices shall be bisected by AF.

[Through E and F draw EG, FG parallel to DF, DE, and join DG meeting EF at H. Use Props. XXIX., XL., Ex. 3, Prop. XXXIII., and Ex. 9, Prop. XXXIV.]

4. If the line joining the vertices of two triangles on equal bases, in the same straight line, but on opposite sides of it, be bisected by the line containing the bases, the triangles are equal.

5. If ABC, DCE be two triangles on equal bases BC, CE, in the same straight line, and if F, G, H, K be the middle points of the sides AB, AC, DC, DE; prove that $FGKH$ is a parallelogram, and that its area is half the difference or half the sum of the areas of the triangles, according as they are on the same side of BE or on opposite sides of it.

6. Prove indirectly that the straight line drawn from the vertex of a triangle to the middle point of the base bisects every parallel to the base.

Hence obtain a method for bisecting a given straight line.

7. Two equal triangles are on equal bases, in the same straight line, and on the same side of it; prove that the intercepts, made by the sides of the triangle on any straight line drawn parallel to the line containing the bases, are equal.

8. On an indefinite line AE three parts AB, BC, CD are cut off, each equal to any given straight line. P is any point without the line AE. Prove that the intercepts made by PA, PB, PC, PD on any line parallel to AE are equal.

Hence give a method for trisecting a given straight line, and for dividing a line into any given number of equal parts.

9. Inscribe a square in a given triangle ABC.

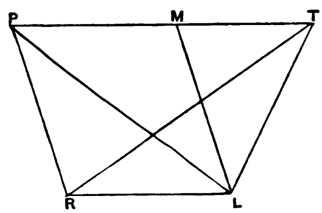
[Let fall AP perpendicular on BC and produce BC to Q making CQ equal to BP; bisect the angle APC by PM meeting AQ at M, through which draw MDE parallel to BC, meeting AC, AB at D, E, which will be angular points of the required square.]

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

Let the ||gram *PRLM* and the $\triangle$ *TRL* be on the same base *RL*, and between the same parallels *RL, PT.*

The ||gram *PRLM* shall be double of the $\triangle$ *TRL*.



Join *PL*. *Post.* 1.

Then the $\triangle$ *PRL* = the $\triangle$ *TRL*,

$\because$ they are on the same base *RL*, and between the same parallels *RL, PT.* I. XXXVII.

But the ||gram *PRLM* is double of the $\triangle$ *PRL*,

$\because$ the diameter *PL* bisects the ||gram. I. XXXIV.

$\therefore$ the ||gram *PRLM* is also double of the $\triangle$ *TRL*.

Wherefore *if a parallelogram &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

I. XXXIV. The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram.

I. XXXVII. Triangles on the same base, and between the same parallels, are equal.

EXERCISES.

1. Prove the converse of this proposition.

2. If a parallelogram and a triangle be on equal bases, and between the same parallels, the parallelogram shall be double of the triangle.

3. A triangle is equal to a parallelogram if they have the same altitude, and the base of the triangle is double that of the parallelogram.

4. If any point be taken within a parallelogram, the sum of the triangles formed by joining the point with the extremities of a pair of opposite sides is equal to half the parallelogram.

5. If two equal straight lines intersect each other at right angles, the quadrilateral formed by joining their extremities is equal to half the square on either straight line.

6. Shew that the area of a triangle is equal to half the product of the base into the altitude.

7. If the bases of two equal triangles be respectively 12 ft. and 14 ft., and the altitude of the first be 7 ft., what is the altitude of the other triangle?

8. The two diagonals of a rhombus are 6 in. and 8 in.; what is the area of the rhombus?

9. Through a point K within a parallelogram $ABCD$ straight lines are drawn parallel to the sides; shew that the difference of the parallelograms of which KA and KC are diagonals is equal to twice the triangle BKD.

10. $ABCD$ is a rectangle, E any point in BC, and F any point in CD. Shew that twice the triangle AEF together with the rectangle two of the sides about an angle of which are equal to BE, DF, is equal to the rectangle $ABCD$.

11. On the two equal sides of a given right-angled isosceles triangle as bases are described two right-angled isosceles triangles; prove that these are together equal to the given triangle.

12. BAC is a right angle in the triangle ABC; on AB, AC are described externally the equilateral triangles ABF, ACE; FA is produced to meet CE at G. Prove that FG bisects CE at right angles, and that each of the triangles BAE, FAC is half the triangle ABC.

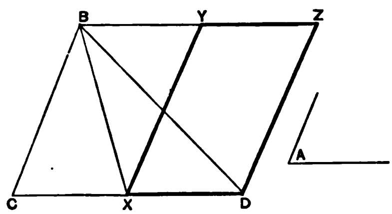
13. ABC is a triangle; D a point in CB such that CD is one-third CB; DA is joined and produced to E so that EA is equal to the half of AD; shew that a parallelogram of which ED and DC are two sides is equal to the triangle ABC.

PROPOSITION XLII. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let BCD be the given $\triangle$, and A the given rectilineal $\angle$.

It is required to describe a ||gram that shall = the given $\triangle BCD$, and have one of its $\angle$s equal to A.



Bisect CD at X; I. x.

at the point X in the st. line DX, make the $\angle DXY$ equal to A; I. XXIII.

through B draw BYZ || to XD, and through D draw DZ || to XY. I. XXXI.

Join BX. *Post.* 1.

Then $DXYZ$ is a ||gram.

And the $\triangle CXB$ = the $\triangle DXB$,

$\because$ they are on equal bases CX, DX, and between the same parallels CD, BZ. I. XXXVIII.

$\therefore$ the $\triangle BCD$ is double of the $\triangle DXB$.

But the ||gram $DXYZ$ is also double of the $\triangle DXB$,

$\because$ they are on the same base DX, and between the same parallels DX, BZ. I. XLI.

$\therefore$ the ||gram $DXYZ$ = the $\triangle BCD$, *Ax.* 6.

and it has one of its $\angle$s DXY equal to the given $\angle A$. *Constr.*

Wherefore *a ||gram $DXYZ$ has been described equal to the given $\triangle BCD$, and having one of its $\angle$s DXY equal to the given $\angle A$.* Q. E. F.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 6. Things which are double of the same thing are equal to one another.

I. x. To bisect a given finite straight line.

I. xxiii. At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

I. xxxi. To draw a straight line through a given point parallel to a given straight line.

I. xxxviii. Triangles on equal bases, and between the same parallels, are equal to one another.

I. xli. If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

EXERCISES.

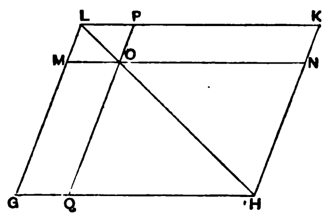
1. Describe a rectangle equal to a given triangle.
2. Describe a rhombus equal to a given triangle.
3. Through two given points in two parallel straight lines draw two lines forming a rhombus with the two parallels.
4. Construct a right-angled triangle equal to a given rhombus.
5. Describe a triangle equal to a given parallelogram, and having an angle equal to a given angle.
6. Describe a parallelogram equal in area and perimeter to a given triangle.
7. Describe a triangle equal to a given triangle, and having a base three times as great.
8. Construct a parallelogram, given the diagonals and a side.
9. Construct a parallelogram, given the diagonals and the angle between them.
10. Construct an equilateral triangle with a given altitude.
11. Through two given points draw two straight lines forming with a straight line given in position an equilateral triangle.
12. Construct a triangle, given the base, one of the base angles, and the sum of the sides.
13. Construct a triangle, having given the base, one of the base angles, and the difference of the sides.
14. Construct an isosceles triangle, whose base and vertical angle are given.
15. Construct a triangle, having given the base, and the sum of a side and the perpendicular from the vertex on the base.
16. Construct a triangle, having given the altitude and the base angles.
17. Inscribe a rhombus in a triangle.
18. The perimeter of an isosceles triangle is greater than the perimeter of a rectangle of the same altitude and area as the triangle.

PROPOSITION XLIII. THEOREM.

The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.

Let $LGHK$ be a ||gram, of which the diameter is LH.
Let MP, QN be ||grams about LH, that is, through which LH passes;
and GO, OK the other ||grams which make up the whole figure $LGHK$, and which are therefore called the complements.

The complement GO shall = the complement OK.



$\because$ $LGHK$ is a ||gram, and LH its diameter,
the $\triangle LGH =$ the $\triangle LKH$. I. XXXIV.

Again, $\because$ $LMOP$ is a ||gram, and LO its diameter,
the $\triangle LMO =$ the $\triangle LPO$. I. XXXIV.

For the same reason, the $\triangle OQH =$ the $\triangle ONH$.

$\therefore$ the 2 $\triangle$s LMO, $OQH =$ the 2 $\triangle$s LPO, ONH. *Ax.* 2.

But the whole $\triangle LGH =$ the whole $\triangle LKH$.

$\therefore$ the remainder, the complement $GO =$ the remainder, the complement OK. *Ax.* 3.

Wherefore *the complements &c.* Q. E. D.

REFERENCES.

Ax. 2. If equals be added to equals the wholes are equal.

Ax. 3. If equals be taken from equals the remainders are equal.

I. xxxiv. The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram.

EXERCISES.

1. The parallelograms about the diameter of a square are squares.

2. If the areas of the squares about the diameter of a square be respectively 4 and 25 sq. inches, find the area of one of the complements.

3. A parallelogram about a diagonal of a rhombus is a rhombus.

4. Shew that the parallelogram LQ in the figure is equal to the parallelogram MK, and the parallelogram GN to the parallelogram PH.

5. If P, Q be two points in the diameter AC of a parallelogram $ABCD$, and if PM, QL, drawn parallel to AB, meet BC, AD in M and L respectively, and if PK, QN, drawn parallel to AD, meet CD, AB in K and N respectively, prove that the parallelograms KL, MN are equal.

6. $ABCD$ is a parallelogram, and E any point in the diagonal AC produced; prove that the triangles EBC, EDC are equal.

7. If through a point O within the parallelogram $ABCD$ two straight lines be drawn parallel to the sides, and the parallelograms OB, OD be equal, the point O is in the diagonal AC.

8. $APOQ$ is a parallelogram; EF, parallel to AP, meets AQ, PO at E, F; and LH, parallel to AQ, meets AP, QO at L, H; LF, EH are produced to meet at C, through which CB, CD are drawn parallel respectively to AP, AQ, to meet AQ, AP at B and D. Prove that O is on the diagonal of the parallelogram $ABCD$.

9. In the figure, the difference of the parallelograms QN and PM is equal to the triangle GOK.

10. In the figure, take any point R in ON, join KR and produce it to meet OQ in S. Through R draw AB parallel to LG, and through S draw CD parallel to LK. Prove that the triangles RLH, RSH are each half of the parallelogram $PQBA$.

11. The sides GL, HK of a quadrilateral $GHKL$ are produced to meet at M, and the sides GH, LK, at N. Prove that the middle points of GK, HL, MN are in a straight line.

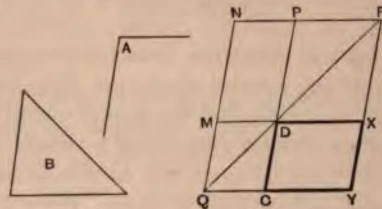
[MN is called the third diagonal of the quadrilateral $GHKL$. Complete the parallelogram $MGOH$, and draw KP parallel to MG and LQ parallel to MH, meeting GO, OH at P, Q. Then PQ passes through N by Ex. 8; MP, GK bisect each other, as also MQ, LH, Ex. 3, Prop. xxxiii.; and the middle points of MP, MQ, MN are in a straight line, Ex. 6, Prop. xxxix.]

PROPOSITION XLIV. THEOREM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let CD be the given st. line, B the given $\triangle$, and A the given rectilineal $\angle$.

It is required to apply to the st. line CD a $\|$gram equal to the $\triangle B$, and having an $\angle$ equal to the $\angle A$.



Make the $\|$gram $DPNM$ equal to the $\triangle B$, and having the $\angle PDM$ equal to the $\angle A$, and so that DP may be in the same st. line with CD; I. XLII.
produce NM to Q; *Post.* 2.
through C draw $CQ \parallel$ to DM or PN, I. XXXI.
and join QD. *Post.* 1.

Then $\because$ the st. line QN falls on the parallels QC, NP,
the $\angle$s CQN, QNP together $= 2$ rt. $\angle$s; I. XXIX.
$\therefore$ the $\angle$s DQN, QNP are together less than 2 rt. $\angle$s.
But st. lines which with another st. line make the intr. $\angle$s on the same side together less than 2 rt. $\angle$s will meet on that side, if produced far enough. *Ax.* 12.
$\therefore$ QD and NP will meet if produced.
Let them be produced and meet at R. *Post.* 2.
Through R draw $RY \parallel$ to PC or NQ, I. XXXI.
and produce QC, MD to meet RY in the points Y, X. *Post.* 2.
Then $QYRN$ is a $\|$gram, of which the diameter is QR; CM, XP are the $\|$grams about QR, and ND, DY are the complements.

$\therefore ND = DY$. I. XLIII.
But $ND =$ the $\triangle B$. *Constr.*
$\therefore DY =$ the $\triangle B$. *Ax.* 1.
And $\because$ the $\angle PDM =$ the $\angle CDX$, I. XV.
and the $\angle PDM =$ the $\angle A$; *Constr.*
$\therefore$ the $\angle CDX =$ the $\angle A$. *Ax.* 1.

Wherefore *to the given st. line CD the $\|$gram DY is applied, equal to the $\triangle B$, and having the $\angle CDX$ equal to the $\angle A$.* Q. E. F.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Post. 2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

I. xv. If two straight lines cut one another, the vertical, opposite angles shall be equal.

I. xxix. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side equal to two right angles.

I. xxxi. To draw a straight line through a given point parallel to a given straight line.

I. xlii. To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

I. xliii. The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

EXERCISES.

1. To a given straight line apply a triangle equal to a given triangle.

2. Describe a rhombus equal to a given rectangle.

3. On a given base construct a rectangle equal in area to a given triangle.

4. To one of the sides of an equilateral triangle apply a parallelogram equal to the triangle and having an angle equal to an angle of the triangle.

5. On the greatest side of a scalene triangle describe a rhombus equal to the triangle.

6. To a given straight line apply a triangle equal to a given parallelogram and having an angle equal to a given angle.

7. On a given base construct an isosceles triangle equal to a given square.

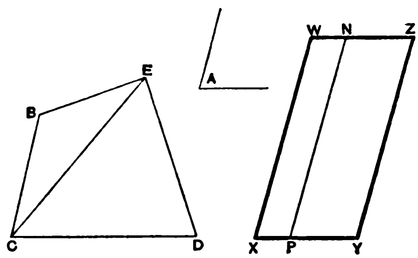
8. From an angular point of a given triangle draw a straight line cutting off from the triangle a given area.

PROPOSITION XLV. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let $BCDE$ be the given rectilineal figure, and A the given rectilineal $\angle$.

It is required to describe a ||gram equal to $BCDE$, and having an $\angle$ equal to A.



Join CE. *Post.* 1.

Describe the ||gram XN equal to the $\triangle BCE$, and having the $\angle WXP$ equal to the $\angle A$. I. XLII.

To the st. line NP apply the ||gram PZ equal to the $\triangle ECD$, and having the $\angle NPY$ equal to the $\angle A$. I. XLIV.

The figure $WXYZ$ shall be the ||gram required.

$\because$ each of the $\angle$s $WXP, NPY =$ the $\angle A$, *Constr.*

the $\angle WXP =$ the $\angle NPY$. *Ax.* 1.

To each add the $\angle XPN$;

$\therefore$ the $\angle$s $WXP, XPN =$ the $\angle$s XPN, NPY. *Ax.* 2.

But the $\angle$s $WXP, XPN = 2$ rt. $\angle$s. I. XXIX.

$\therefore$ the $\angle$s $XPN, NPY = 2$ rt. $\angle$s. *Ax.* 1.

And $\because$ at the point P in the st. line PN, the 2 st. lines XP, PY, on the opposite sides of it, make the adjacent $\angle$s equal to 2 rt. $\angle$s,

$\therefore$ XP is in the same st. line with PY. I. XIV.

And $\because$ the st. line PN meets the parallels XY, WN,

the $\angle YPN =$ the alternate $\angle PNW$. I. XXIX.

To each add the $\angle PNZ$;

$\therefore$ the $\angle$s $YPN, PNZ =$ the $\angle$s PNW, PNZ. *Ax.* 2.

But the $\angle$s $YPN, PNZ = 2$ rt. $\angle$s. I. XXIX.

$\therefore$ the $\angle$s $PNW, PNZ = 2$ rt. $\angle$s. *Ax.* 1.

$\therefore$ WN is in the same st. line with NZ. I. XIV.

And $\because$ XW is || to PN, and PN || to YZ, *Constr.*

$\therefore$ XW is || to YZ, I. XXX.

and XY is || to WZ.

$\therefore$ $WXYZ$ is a ||gram.

And $\because$ the $\|$gram $XN =$ the $\triangle BCE$, *Constr.*
and the $\|$gram $PZ =$ the $\triangle ECD$; *Constr.*
$\therefore$ the whole $\|$gram $WXYZ =$ the whole rectilineal figure $BCDE$. *Ax.* 2.

Wherefore *the $\|$gram $WXYZ$ has been described equal to the given rectilineal figure $BCDE$, and having the $\angle$ WXY equal to the given $\angle$ A.* Q. E. F.

Corollary. From this it is manifest how, to a given straight line to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; namely, by applying to the given straight line a parallelogram equal to the first triangle BCE, and having an angle equal to the given angle; and so on. I. XLIV.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 2. If equals be added to equals the wholes are equal.

I. XIV. If, at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

I. XXIX. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

I. XXX. Straight lines which are parallel to the same straight line are parallel to each other.

I. XLII. To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

I. XLIV. To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

EXERCISES.

1. To a given straight line apply a rectangle equal to a given parallelogram.

2. On a given base construct a rectangle equal to a given square.

3. Describe a parallelogram equal to a given parallelogram and having a given altitude.

4. Construct a rectangle equal to the sum of two given rectilineal figures.

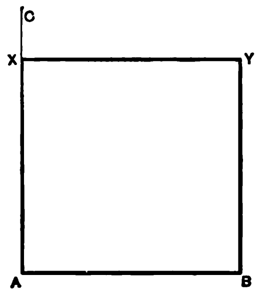
5. Construct a rectangle equal to the difference of two given rectilineal figures.

PROPOSITION XLVI. PROBLEM.

To describe a square upon a given straight line.

Let AB be the given st. line.

It is required to describe a square on AB.



From the point A draw AC at rt. $\angle$s to AB; I. XI.
make AX equal to AB; I. III.
through X draw XY $\parallel$ to AB, and through B draw BY $\parallel$ to AX, meeting XY in Y. I. XXXI.

Then $AXYB$ is a $\parallel$gram;

$\therefore$ $AX = BY$, and $AB = XY$. I. XXXIV.

But $AX = AB$. *Constr.*

$\therefore$ the 4 st. lines BA, AX, XY, YB are equal to one another.

$\therefore$ the $\parallel$gram $AXYB$ is equilateral.

Likewise all its $\angle$s are rt. $\angle$s.

For, $\because$ the st. line AX meets the parallels AB, XY,

the $\angle$s BAX, AXY together $= 2$ rt. $\angle$s. I. XXIX.

But BAX is a rt. $\angle$. *Constr.*

$\therefore$ also AXY is a rt. $\angle$. *Ax.* 3.

And the opposite $\angle$s of $\parallel$grams are equal; I. XXXIV.

$\therefore$ each of the opposite $\angle$s XYB, YBA is a rt. $\angle$. *Ax.* 1.

$\therefore$ the figure $AXYB$ is rectangular;

and it has been shewn to be equilateral.

$\therefore$ *it is a square;* *Def.* 30.

and it is described on the given st. line AB. Q. E. F.

Corollary. Hence, every parallelogram that has one of its angles a right angle, has all its angles right angles.

REFERENCES.

Def. 30. A square is a four-sided figure which has all its sides equal, and all its angles right angles.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 3. If equals be taken from equals the remainders are equal.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. XI. To draw a straight line at right angles to a given straight line, from a given point in the same.

I. XXIX. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

I. XXXI. To draw a straight line through a given point parallel to a given straight line.

I. XXXIV. The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram.

EXERCISES.

1. Construct a rectangle whose sides are equal to two given lines.

2. Prove by superposition that all rectangles, which have two sides respectively equal to two given straight lines, are equal.

[Hence a rectangle having sides respectively equal to two given straight lines is called the rectangle contained by those lines.]

3. Construct a square having given a diagonal.

4. If two squares are equal, shew that a side of one is equal to a side of the other.

5. If in the sides of a square at equal distances from the angular points four other points be taken in order, one on each side; the figure contained by the straight lines which join them shall also be a square.

6. $ABCD$ is a square; in AB take any point E, and from AD, CD, CB cut off AF, CG, CH, each equal to AE. $EFGH$ is a rectangle.

7. Within a square $ABCD$ a square $EFGH$ is inscribed; prove that the sides of $ABCD$ are equally divided in the points E, F, G, H.

8. Divide a square into four right-angled triangles and a square.

9. If the middle points of opposite sides of a square be joined, prove that the square is thus divided into four equal squares.

10. The square on a line is four times the square on half the line.

11. If straight lines be drawn through the angular points of a square and parallel to the diagonals; shew that these straight lines themselves form a square.

12. On the sides of a square are described equilateral triangles all external; shew that the straight lines joining their vertices form a square.

13. $ABCD$ is a square of which the diagonal AC is equal to $7\sqrt{2}$ inches. In AC take a point E, such that AE is equal to $2\sqrt{2}$ inches. Find the areas of the whole square and of the four figures into which the square is divided by straight lines through E parallel to the sides.

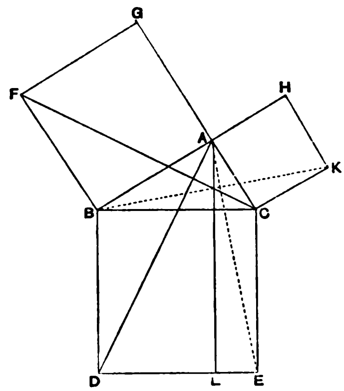
14. If a square and an equilateral triangle be described upon the same straight line, the square is greater than twice the triangle.

15. A square $ABCD$ and an equilateral triangle ABE are described on the same base AB and on opposite sides of it. If F the middle point of AE be joined with C, and AE be produced to G such that EG is equal to FB; shew that GB, FC will be equal.

PROPOSITION XLVII. THEOREM.

In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Let ABC be a right-angled $\triangle$, having the rt. $\angle$ BAC.
The sq. on BC shall = the sqq. on BA, AC.



On BC, CA, AB describe the sqq. $BDEC$, CH, BG. I. XLVI.
Through A draw $AL \parallel$ to BD or CE; I. XXXI.
and join AD, FC. *Post.* 1.
Then $\because$ BAC is a rt. $\angle$, *Hyp.*
and BAG is also a rt. $\angle$; *Def.* 30.
$\therefore$ the 2 st. lines AC, AG, on the opposite sides of AB, make with it at the point A the adjacent $\angle$ s equal to 2 rt. $\angle$ s;
$\therefore$ CA is in the same st. line with AG. I. XIV.
For the same reason BA, AH are in the same st. line.
Now the $\angle DBC$ = the $\angle FBA$, for each of them is a rt. $\angle$. *Ax.* 11.
To each add the $\angle ABC$.
$\therefore$ the whole $\angle DBA$ = the whole $\angle FBC$. *Ax.* 2.
And $\because$ the 2 sides AB, BD = the 2 sides FB, BC, each to each, *Def.* 30.
and the $\angle ABD$ = the $\angle FBC$;
$\therefore$ the $\triangle$ ABD = the $\triangle$ FBC. I. IV.
Now the $\parallel$gram BL is double of the $\triangle$ ABD, for they are on the same base BD, and between the same parallels BD, AL. I. XLI.
And the sq. BG is double of the $\triangle$ FBC, for they are on the same base FB, and between the same parallels FB, GC. I. XLI.
But the doubles of equals are equal to one another. *Ax.* 6.
$\therefore$ the $\parallel$gram BL = the sq. BG.
In the same manner, by joining AE, BK, it may be shewn, that the $\parallel$gram CL = the sq. CH.
$\therefore$ the whole sq. $BDEC$ = the 2 sqq. BG, CH. *Ax.* 2.
And the sqq. $BDEC$, BG, CH are described on BC, CA, AB.
$\therefore$ *the* sq. on BC = the sqq. on BA, AC.
Wherefore in any right-angled triangle &c. Q. E. D.

REFERENCES.

Def. 30. A square is a four-sided figure which has all its sides equal, and all its angles right angles.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 2. If equals be added to equals the wholes are equal.

Ax. 6. Things which are double of the same thing are equal to one another.

Ax. 11. All right angles are equal to one another.

I. IV. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

I. XIV. If, at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

I. XXXI. To draw a straight line through a given point parallel to a given straight line.

I. XLI. If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

I. XLVI. To describe a square on a given straight line.

EXERCISES.

1. The sum of the squares on the diagonals of a rectangle is equal to the sum of the squares on the four sides.

2. The squares on the sides of a rhombus are together equal to the squares on the diagonals.

3. Each of the equal sides, AB, AC, of an isosceles triangle ABC is 5 ft., and the base is 6 ft. If AD bisect BC in D, find AB.

4. If the area of a square be 50 sq. inches, what will be the area of another square having the diagonal of the first square as a side?

5. If the perpendiculars from the extremities of the base of a triangle on the opposite sides be equal, the triangle is isosceles.

6. In the triangle ABC, BD is drawn perpendicular to AC. Prove that the difference of the squares on AD, CD is equal to the difference of the squares on AB, BC.

7. The equilateral triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the equilateral triangles described on the sides containing the right angle.

8. The hypotenuses of three isosceles right-angled triangles form a right-angled triangle. Prove that one of the isosceles triangles is equal to the sum of the other two.

9. The bisectors of the angles of any triangle are concurrent.

10. If a square be described on the diagonal of a square, and another square on the diagonal of this, and so on; the last square so described will be equal to the sum of all the others so described, together with twice the original square.

11. The sides AB, BC, CA of the triangle ABC are respectively parallel to the sides DE, EF, FD of the triangle DEF; BG, BH are drawn perpendiculars to EF, DE; CK, CL, to DF, EF; AM, AN, to DF, DE. Prove that the sum of the squares on AG, BK, CN is equal to the sum of the squares on AL, BM, CH.

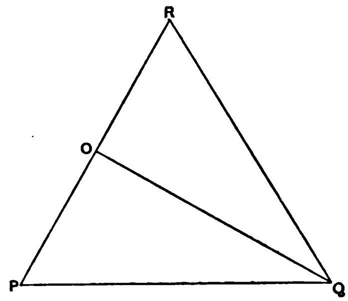
12. Squares are described on the sides of an acute-angled triangle; a perpendicular is let fall from each of its angular points on the opposite side and produced to divide the square on that side into two parts. Prove that the six rectangles into which the squares are divided are equal two and two.

PROPOSITION XLVIII. THEOREM.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

Let the sq. on PQ, one of the sides of the $\triangle\ OPQ$ = the sqq. on the other sides PO, OQ.

The $\angle\ POQ$ shall be a rt. $\angle$.



From the point O draw OR at rt. $\angle$s to OQ; I. XI.
make $OR = OP$, I. III.
and join RQ. *Post.* 1.
$\because OR = OP$, *Constr.*
the sq. on OR = the sq. on OP.
To each add the sq. on OQ.
$\therefore$ the sqq. on RO, OQ = the sqq. on PO, OQ. *Ax.* 2.
But the sqq. on RO, OQ = the sq. on RQ, I. XLVII.
$\because ROQ$ is a rt. $\angle$. *Constr.*
And the sqq. on PO, OQ = the sq. on PQ. *Hyp.*
$\therefore$ the sq. on PQ = the sq. on RQ. *Ax.* 1.
$\therefore$ the side PQ = the side RQ.
And $\because PO = RO$, *Constr.*
and OQ is common to the 2 $\triangle$s POQ, ROQ;
the 2 sides PO, OQ = the 2 sides RO, OQ, each to each;
and the base PQ has been shewn = the base RQ;
$\therefore$ the $\angle\ POQ$ = the $\angle\ ROQ$. I. VIII.
But ROQ is a rt. $\angle$: *Constr.*
$\therefore POQ$ is a rt. $\angle$. *Ax.* 1.

Wherefore *if the square &c.* Q. E. D.

REFERENCES.

Post. 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Ax. 1. Things which are equal to the same thing are equal to one another.

Ax. 2. If equals be added to equals the wholes are equal.

I. III. From the greater of two given straight lines to cut off a part equal to the less.

I. VIII. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

I. XI. To draw a straight line at right angles to a given straight line, from a given point in the same.

I. XLVII. In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

EXERCISES.

1. Shew that the following are some numbers which may represent the sides of right-angled triangles :— 3, 4, 5 ; 5, 12, 13 ; 6, 8, 10 ; 7, 24, 25 ; 8, 15, 17 ; 9, 40, 41 ; 10, 24, 26 ; 11, 60, 61 ; 12, 16, 20.

2. Taking one of the sets of numbers in Ex. 1., shew how to make the corner of a lawn-tennis court square.

3. A rectangle, one of whose sides is 1 ft., is equal to a triangle whose sides are 5, 12, 13 ft.; find the other side.

4. If the square on one side of a triangle be greater than the sum of the squares on the other two sides, the angle contained by these two sides is obtuse ; and if less, acute.

5. Classify the triangles whose sides are :— (*a*) 16, 63, 65 ; (*b*) 13, 88, 84 ; (*c*) 8, 15, 16.

6. HKL is a triangle having KL equal to 25 inches, HK to 24 inches, and HL to 7 inches ; KL is bisected at F, and HF is produced to M so that FM is equal to HF. Prove that $HKML$ is a rectangle.

7. ABC is a triangle ; D, the middle point of BC, is joined to A : prove that if the squares on BA, AC are together equal to the square on twice AD, the angle BAC is a right angle.

8. One end of a ladder 12 ft. long rests on a ledge in a wall, and the other end is fastened by a rope 20 ft. long to a ring fixed 16 ft. above the ledge. What is the position of the ladder?

9. Two men, A and B, set out from the same place, A walking in the N.E. direction 7 miles, whilst B drives 24 miles, after which they are 25 miles apart. In what direction did B drive?

10. If the equilateral triangle described on one of the sides of a triangle be equal to the sum of the equilateral triangles described on the other two sides of it, the angle contained by these two sides is a right angle.

PLANE LOCI.

If all the points on one or more lines, straight or curved, and no other points, satisfy given geometrical conditions, the line or lines form what is called the *locus* of the points satisfying the conditions.

If X be the locus of points satisfying certain given conditions, and Y be the locus of points satisfying other given conditions, then the point or points of intersection of X and Y, and no other points, satisfy all the given conditions.

If a geometrical magnitude, such as an angle, a line, or an area, can be expressed in terms of a magnitude, which is fixed or given; or, if it keep the same value throughout while changing its position gradually according to a given law; it is said to be a *constant*. For example, the sum of the three angles of a triangle is constant (I. XXXII.); if a circle be given, its radius is constant (Def. 15); if the base of a triangle be a given line, and the vertex be any point on a given line parallel to the base, the area of the triangle is constant (I. XXXVII.).

EXERCISES.

1. Find the locus, in a fixed plane, of
   (i) Points at a given distance from a fixed point.
   (ii) Points at a given distance from a fixed circle.
   (iii) Points at a given distance from a given straight line.
   (iv) Points equally distant from two given points.
   (v) Points equidistant from two given intersecting lines.
   (vi) The vertices of equal triangles, which are on the same base, and on the same side of it.
   (vii) The points of bisection of lines drawn from a given point to meet a given straight line.

2. From a given point, draw to meet a given straight line, a straight line of given length.

3. From the vertex of a scalene triangle draw a straight line to the base, which shall exceed the less side by as much as it is exceeded by the greater.

4. Find a point such that the perpendiculars let fall from it on two given straight lines shall be respectively equal to two given straight lines. Shew that there are four such points.

5. Construct a triangle, having given two sides and the angle opposite one of them.

6. Find a square equal to the difference of two given squares.

7. Draw a straight line equal to one straight line, parallel to another, and terminated by two other given straight lines.

8. Find the locus of a point, the sum of whose distances from two fixed lines is constant.

9. Find the locus of a point, the difference of whose distances from two fixed lines is constant.

10. From any point within an equilateral triangle the sum of the perpendiculars on the three sides is constant.

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EXERCISES.

1. Construct a right-angled isosceles triangle having given the hypotenuse.

2. Inscribe a square in a given right-angled triangle.

3. Through a given point draw a straight line such that the part of it intercepted between two given parallel straight lines may be equal to a given line.

4. Two straight lines AB, AC are given in position. Find in AB a point P such that a perpendicular being drawn from it to AC, AP may exceed this perpendicular by a given length.

5. ABC is a triangle. Draw a straight line of given length perpendicular to BC and terminated by AC, CB.

6. BAC is the greatest angle of the triangle ABC. Divide it into two parts whose difference shall be equal to the angle ABC.

7. Construct a right-angled isosceles triangle having given: (i) the sum of the hypotenuse and one side; (ii) the difference between the hypotenuse and one side.

8. Construct an isosceles triangle, whose vertical angle is equal to a given angle, and whose base is equal to a given line.

9. AB, AC are two given straight lines, and P a given point in AB. Draw through P a straight line to meet AC at Q so that the angle APQ may be three times the angle AQP.

10. Divide a straight line into two parts such that the square on one part may be double the square on the other.

11. Find a point in a given straight line at a given distance from a given straight line.

12. P is any point in AB, a side of the triangle ABC. Draw from P a straight line to meet BC in Q and AC produced in R, such that PQ may be equal to QR.

13. Three straight lines AB, AC, AD meet at A. Draw a straight line cutting them so that the two parts intercepted may be equal.

EXERCISES (*continued*).

14. From a given point without an angle BAC draw a straight line such that the intercept between the point and the nearest line may be equal to the intercept between the two lines.

15. Find a point D in the hypotenuse AB of a right-angled triangle ABC, such that DB may be equal to the perpendicular from D on AC.

16. In the base of a triangle find a point from which lines drawn parallel to the sides of the triangle and terminated by them are equal.

17. ABC is a triangle in which C is a right angle. Draw a straight line parallel to a given straight line, so as to be terminated by CA and CB and bisected by AB.

18. Describe a triangle, whose base shall be three times that of a given triangle and whose area shall be equal to that of the given triangle.

19. Construct a triangle, having given two sides and the area.

20. In two parallel straight lines find two points equidistant from a given point, and such that the line joining them is parallel to a given line.

21. Through four given points draw four straight lines which shall form a square.

[See Ex. 16, p. 101.]

22. Draw a line parallel to a side of a triangle so that the intercept on it by the other two sides may be equal to (i) the sum, (ii) the difference of the segments of the two sides adjacent to the first side.

23. Construct a square equal to the sum of (*a*) two, (*b*) three given squares.

24. Describe a parallelogram having given a side, an angle, and a diagonal.

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